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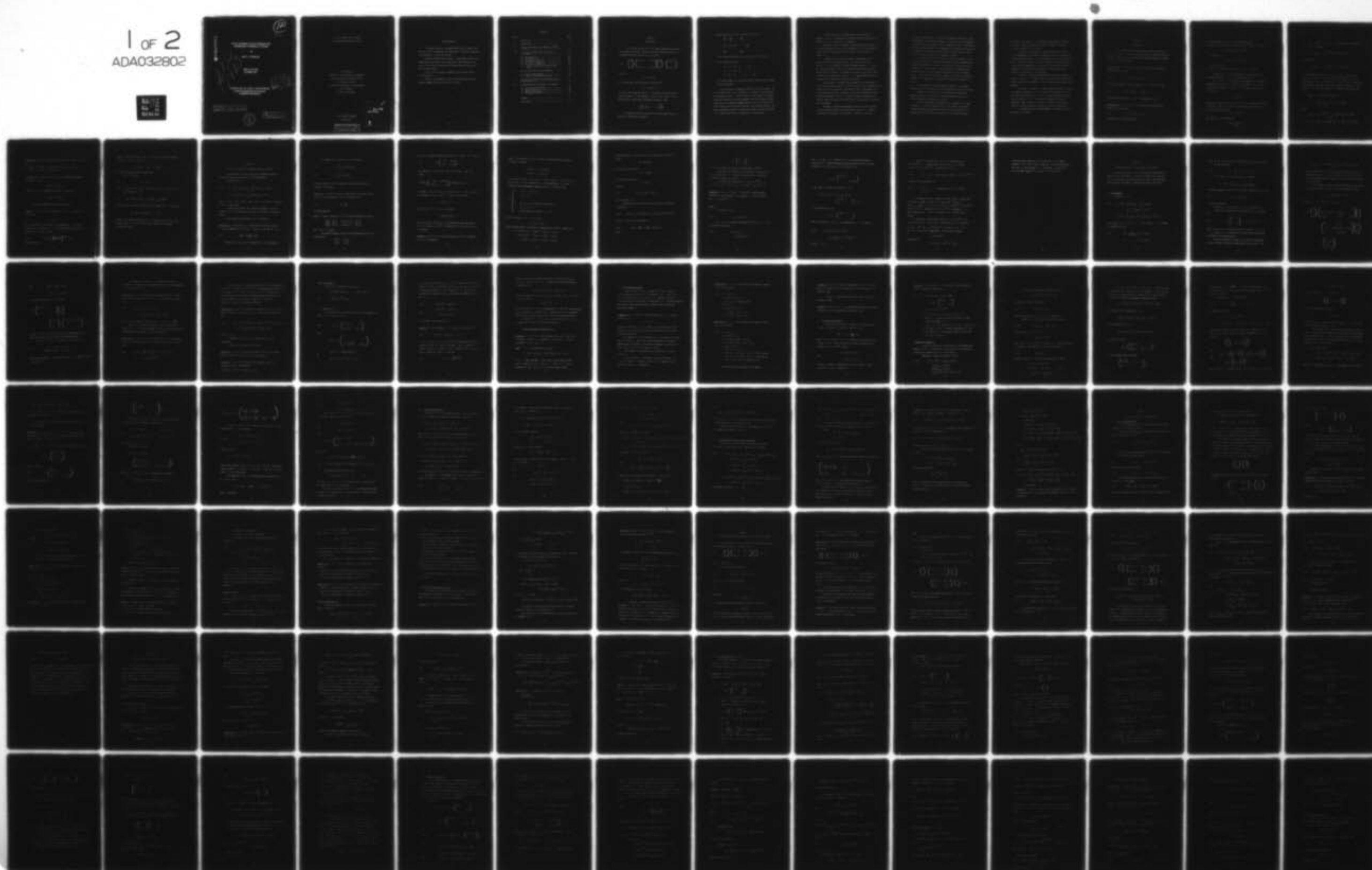

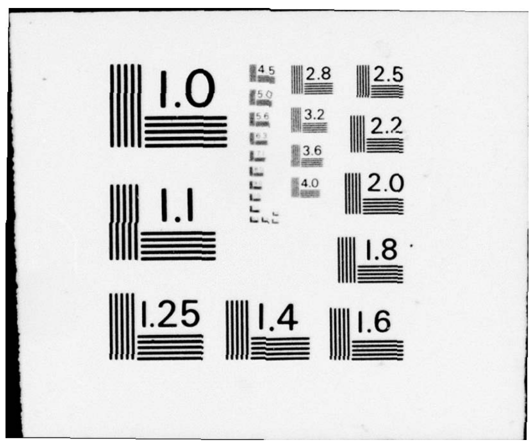
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INITIAL BOUNDARY VALUE PROBLEMS FOR INCOMPLETELY PARABOLIC SYSTEMS

by

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Contents

CHAPTER I

INTRODUCTION

In this thesis we will treat the initial boundary value problem for incompletely parabolic systems of partial differential equations. Incompletely parabolic systems are of the form

$$(1.1) \qquad \begin{pmatrix} w_1 \\ \vdots \\ w_r \end{pmatrix}_t = \begin{pmatrix} P(x,t,D) & A(x,t,D) \\ & \\ B(x,t,D) & Q(x,t,D) \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_r \end{pmatrix} + \begin{pmatrix} f_1(x,t) \\ \vdots \\ f_r(x,t) \end{pmatrix} ,$$

such that

$$u_t = P(x,t,D)u$$

is a second order Petrovskii parabolic system, and

$$v_t = Q(x,t,D)v$$

is a first order hyperbolic system. (The precise definitions will be given in Chapter II.) The operators $A(x,t,D)$ and $B(x,t,D)$ can be arbitrary first order linear differential operators, where

$$D = \left(-i \frac{\partial}{\partial x_0} , \ldots , -i \frac{\partial}{\partial x_n} \right) .$$

Incompletely parabolic systems arise in many applications, we present the following two examples.

1) Coupled Sound and Heat Flow (Richtmyer [16, p. 170]).

$$\frac{\partial e}{\partial t} = \sigma \frac{\partial^2 e}{\partial x^2} \qquad - c \frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial t} = c(1-\gamma) \frac{\partial e}{\partial x} \qquad + c \frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial t} = \qquad c \frac{\partial u}{\partial x}$$

For the higher dimensional equations see Pyasta [14 , p. 210].

2) Viscous Shallow Water Flow.

$$u_t = \mu\Delta u - uu_x - vu_y \qquad - \varphi_x$$

$$v_t = \mu\Delta v - uv_x - vv_y \qquad - \varphi_y$$

$$\varphi_t = -\varphi u_x - \varphi v_y \qquad - u\varphi_x \qquad - v\varphi_y$$

This second example is a non-linear system; however we will consider only linear systems.

The literature on incompletely parabolic systems is sparse. The first example above, due to Richtmyer [16], appears to have stimulated the subsequent research. Novik [13] and Lions and Raviart [9] proved existence theorems for the Cauchy problem using very different methods. Novik used finite difference approximations to prove the existence of a solution while Lions and Raviart used functional analytic methods. The name "incompletely parabolic" comes from the paper by Belov and Yanenko [3], in which they discuss the smoothness of the solution.

In all of the above, the leading symbol of the parabolic operator $P(x,t,D)$ was assumed to be negative definite and the leading symbol of $Q(x,t,D)$ was taken to be symmetric.

Difference schemes for Example 1 are discussed by Richtmyer [16] (see also Richtmyer and Morton [17], and Morimoto [11]), and difference schemes for other systems are given by Pyasta [14] and Lions and Raviart [9].

It appears that the initial boundary value problem has not been treated in any generality before. The results of Lions and Raviart [9] can be applied to coercive boundary conditions. We know of no other treatment of the initial boundary value problem for incompletely parabolic systems. Our approach is similar to that which Kreiss used for strictly hyperbolic systems [6]. In unpublished work, Kreiss has applied the same method to parabolic equations.

We have chosen to consider only problems that are well-posed in the L^2 norm for several reasons. The first reason is that the determination of whether or not the problem is well-posed in the L^2 norm depends only on the highest order terms. Moreover, one need only consider the frozen coefficient problems, that is, the constant coefficient problems that arise by fixing the values of the coefficients at each point in the domain being considered.

Secondly, in computing approximate solutions to partial differential equations by means of finite difference schemes, it is important to establish the stability of the method. (Stability is the exact analogue of well-posedness.) For variable coefficient difference schemes there appears to be no general approach to stability apart from examining the frozen coefficient problem. But this approach can not be valid for the difference equations unless it also applies to the differential equations. It is important therefore to distinguish that class of partial differential equations for which such an approach is valid.

The chief results of this thesis are stated in Theorems 6.1 and 6.2. They show that one can determine whether an initial boundary value problem is well-posed by checking certain algebraic conditions at each boundary point. These algebraic conditions arise from examining the frozen coefficient initial boundary value problems for each boundary point. The necessary and sufficient conditions for constant coefficient problems on a half-space to be well-posed are stated in Theorems 4.5 and 5.1.

We now briefly outline the remaining chapters. In Chapter II we define most of the terms and notations we will be using in this thesis.

Chapter III deals with the Cauchy problem for incompletely parabolic systems. We establish an energy inequality and use it to prove existence and uniqueness of solutions.

The initial boundary value problem on a half-space is the topic of Chapter IV. We first consider the case with constant coefficients obtaining necessary and sufficient conditions for the problem to be well-posed in an L^2 norm with scaling factors. The scaling factors are needed to accommodate both the parabolic and hyperbolic behavior. Then using a Gårding inequality we extend the results to variable coefficient problems on a half-space.

In Chapter V we introduce another scaling of the L^2 norm which is also appropriate for incompletely parabolic systems on a half-space. Many results are analogous to those of Chapter IV and are stated without proof. In addition we show the relation between the various scalings. For the sake of completeness we also state the necessary and sufficient conditions for the parabolic and hyperbolic initial boundary value problems on a half-space to be well-posed.

The initial boundary value problem on bounded smooth domains is treated in Chapter VI. We apply the results of the Cauchy and half-plane initial boundary value problems to obtain a sufficient condition for well-posedness.

In Chapter VII we present a theory of pseudo-differential operators that depend on a parameter. We then construct the symmetrizer $R(\omega,s)$ which was used in Chapter IV. $R(\omega,s)$ is a pseudo-differential operator with the parameter $\mathrm{Re}(s)$. We then prove analogues of Gårding's inequality. These are used in Chapter IV to extend the results for constant coefficient equations to those with variable coefficients.

In an appendix we give several examples to illustrate the method presented in this thesis.

CHAPTER II

PRELIMINARIES

In this chapter we define many of the terms and the notations that will be used. In the first chapter we defined incompletely parabolic systems in terms of Petrovskii parabolic systems and hyperbolic systems. We now state the definitions.

Definition 2.1. If $P(x,t,D)$ is a second order linear differential operator, then the system

$$u_t = P(x,t,D)u + f(x,t) \tag{2.1}$$

is Petrovskii parabolic if the eigenvalues λ of $P(x,t,\xi)$ satisfy

$$\operatorname{Re}\lambda \leq -\alpha(\xi_0^2 + \cdots + \xi_n^2) + \beta \tag{2.2}$$

for some positive constants α and β independent of (x,t).

Definition 2.2. If $Q(x,t,D)$ is a first order linear differential operator, then the system

$$v_t = Q(x,t,D)v + f(x,t) \tag{2.3}$$

is hyperbolic if the following hold.

1. The eigenvalues of $Q(x,t,\xi)$ are purely imaginary.

2. There is a continuous matrix valued function $T(x,t,\xi)$ such that

$$T(x,t,\xi)\, Q(x,t,\xi)\, T(x,t,\xi)^{-1}$$

is diagonal and

$$\|T(x,t,\xi)\|,\ \|T(x,t,\xi)^{-1}\| \le K$$

for some constant K.

3. $T(x,t,\xi)$ is as differentiable in (x,t) as is $Q(x,t,\xi)$.

Equation (2.3) is strictly hyperbolic if all eigenvalues of the leading symbol of $Q(x,t,\xi)$ are distinct for $\xi \neq 0$.

For a vector a, a' will denote the transpose and the conjugate transpose will be denoted a^t, similarly for a matrix A, A^t will be its conjugate transpose. If $(P_i)_{i=1}^n$ is an n-tuple of matrices and $\omega \in \mathbb{R}^n$, then

$$P\cdot\omega = \sum_{i=1}^{n} P_i\omega_i \quad .$$

Similarly for vectors $x, \xi \in \mathbb{R}^n$, $x\cdot\xi$ will be the standard inner product. For vectors u and v in $\mathbb{C}^n$ the inner product will be written $u^t v$ and the norm will be given by

$$|u| = (u^t u)^{1/2}.$$

For a matrix A the norm will be

$$\|A\| = \sup_{|u|=1} |Au| \ .$$

If $\Omega \subseteq \mathbb{R}^{n+1}$ and u and v are measurable vector functions on $\Omega \times [0,\infty)$, we define

$$(u,v)_{\eta,\Omega} = \int_0^\infty \int_\Omega e^{-2\eta t}\, u(x,t)^t\, v(x,t)\, dx\, dt$$

and, similarly,

$$\langle u,v\rangle_{\eta,\partial\Omega} = \int_0^\infty \int_{\partial\Omega} e^{-2\eta t}\, u(x,t)^t\, v(x,t)\, dx\, dt .$$

The corresponding norms are $\|u\|_{\eta,\Omega}$ and $|u|_{\eta,\partial\Omega}$, respectively. We shall use $\|w\|_{r,\Omega}$ to denote the norm of the function w in the Sobolev space $H^r(\Omega)$. The subscripts "η" and "r" on the norms have quite different meanings, but this should cause no confusion. We will drop the "Ω" and "$\partial\Omega$" whenever it is clear from the context what Ω is.

When $\Omega = \mathbb{R}^n$ we have the relation

$$\|u\|_\eta^2 = \int_{-\infty}^\infty \int_{\mathbb{R}^n} |\hat{u}(\omega,\, \eta + i\tau)|^2\, d\omega\, d\tau$$

where

$$\hat{u}(\omega,s) = (2\pi)^{-(n+1)/2} \int_0^\infty \int_{\mathbb{R}^n} e^{-st}\, e^{-i\omega\cdot x}\, u(x,t)\, dx\, dt .$$

For $f \in L^2(\mathbb{R}_+)$ we define $\|f\|_+$ by $\|f\|_+^2 = \int_0^\infty |f(x)|^2\, dx$.

Definition 2.3. We say a function $f(x)$ tends to a constant as $|x| \to \infty$ if

1) there is a constant f_∞ such that $f(x) \to f_\infty$ as $|x| \to \infty$

2) $D^\alpha f(x) \to 0$ as $|x| \to \infty$ for all $\alpha > 0$.

Definition 2.4. We say a function $h(\omega,s)$ has parabolic homogeneity of degree r if for $\rho > 0$

$$h(\rho\omega,\rho^2 s) = \rho^r h(\omega,s) .$$

Similarly $h(\omega,s)$ has hyperbolic homogeneity of degree r if

$$h(\rho\omega,\rho s) = \rho^r h(\omega,s) .$$

An algebraic lemma we will need several times is

Lemma 2.1. If A is an $a \times a$ matrix and B is a $b \times b$ matrix, then the equation

$$AX - XB = C \tag{2.5}$$

has a unique solution if and only if no eigenvalue of A is also an eigenvalue of B. Moreover if δ is the minimum distance between the eigenvalues of A and B, we have

$$\|X\| \le k\, \delta^{-1} \left(\frac{\|A\| + \|B\|}{\delta} \right)^{a+b-1} \|C\| \tag{2.6}$$

for some constant k.

Proof. By Schur's Theorem (Jacobson [4], page 193) there are orthogonal matrices O_1 and O_2 such that

$$\tilde{A} = O_1^t A O_1 \qquad \text{and} \qquad \tilde{B} = O_2^t B O_2$$

are lower and upper triangular, respectively.

Then (2.5) becomes

$$\tilde{A}Y - Y\tilde{B} = \tilde{C} \tag{2.7}$$

where $Y = O_1^t X O_2$ and $\tilde{C} = O_1^t C O_2$. Note that $\|Y\| = \|X\|$ and $\|\tilde{C}\| = \|C\|$.

We can rewrite (2.7) as

$$(\tilde{a}_{ii} - \tilde{b}_{kk}) y_{ik} = \tilde{c}_{ik} - \sum_{j<i} \tilde{a}_{ij} y_{jk} + \sum_{j<k} y_{ij} \tilde{b}_{jk} . \tag{2.8}$$

Equation (2.8) is a recursive formula for the elements of Y in the order

$$y_{11},\ y_{12},\ y_{21},\ y_{13},\ y_{22},\ \cdots\ ,\ y_{ab} \ .$$

Equation (2.6) follows easily from (2.8). We also have that if $\tilde{a}_{ii} = \tilde{b}_{kk}$ for some indices i and k, we lack either existence or uniqueness. This proves the lemma.

CHAPTER III

THE CAUCHY PROBLEM FOR INCOMPLETELY PARABOLIC SYSTEMS

We now consider the Cauchy Problem for incompletely parabolic systems and give some conditions for its well-posedness.

We rewrite (1.1) as

$$u_t = \sum_{i,j=0}^{n} P_{ij}(x,t)u_{x_i x_j} + \sum_{k=0}^{n} A_k(x,t)v_{x_k} + F_1(x,t) \tag{3.1}$$

$$v_t = \sum_{k=0}^{n} B_k(x,t)u_{x_k} + \sum_{k=0}^{n} Q_k(x,t)v_{x_k} + F_2(x,t)$$

where $w = (u,v)'$, $w_0(x) = w(x,0)$ and u and v are vectors of dimension p and q, respectively.

The lower order terms in (1.1) have been dropped in (3.1) for simplicity, but it is easily seen that they do not affect the subsequent results.

We seek conditions under which (3.1) is well-posed.

Definition 3.1. The system (3.1) is well-posed if there are constants C and η_0, independent of w_0 and $F = (F_1,F_2)'$ such that for $\eta \geq \eta_0$

$$\|w\|_\eta^2 \leq C(\|w_0\|_0^2 + \|F\|_\eta^2) . \tag{3.2}$$

Equation (3.1) is said to be ill-posed if it is not well-posed.

We consider (3.1) in relation to the two systems

(3.3)
$$u_t = \sum_{i,j=0}^{n} P_{ij}(x,t)u_{x_i x_j}$$

(3.4)
$$v_t = \sum_{k=0}^{n} Q_k(x,t)v_{x_k} .$$

We have assumed that (3.4) is hyperbolic, we now show that such an assumption is necessary.

Theorem 3.1. If (3.1) has constant coefficients, a necessary condition for (3.1) to be well-posed is that the eigenvalues of

(3.5)
$$i \sum_{k=0}^{n} Q_k \xi_k$$

be purely imaginary.

Proof. We Fourier transform (3.1) in the spatial variables and obtain

$$\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}_t = \begin{pmatrix} P(\xi) & iA\cdot\xi \\ iB\cdot\xi & iQ\cdot\xi \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} + \begin{pmatrix} \hat{F}_1 \\ \hat{F}_2 \end{pmatrix}$$

where $P(\xi) = -\sum P_{ij}\xi_i\xi_j$.

A well-known necessary condition for well-posedness is that the eigenvalues of

$$\begin{pmatrix} P(\xi) & iA\cdot\xi \\ iB\cdot\xi & iQ\cdot\xi \end{pmatrix}$$

have their real parts bounded above for all ξ. Now let $\lambda(\xi)$ satisfy

$$(3.6) \qquad \det\begin{pmatrix} \lambda - P(\xi) & -iA\cdot\xi \\ -iB\cdot\xi & \lambda - iQ\cdot\xi \end{pmatrix} = 0 .$$

If we replace ξ by $\beta^{-1}\xi$ and $\lambda(\xi)$ by $\beta^{-1}\lambda'(\xi,\beta)$ then (3.6) becomes

$$\beta^{-(2p+q)}\det\begin{pmatrix} \beta\lambda' - P(\xi) & -i\beta^{1/2}A\cdot\xi \\ -i\beta^{1/2}B\cdot\xi & \lambda' - iQ\cdot\xi \end{pmatrix} = \beta^{-(2p+q)}p(\lambda',\xi,\beta) = 0 .$$

Now suppose a root of $p(\lambda',\xi,0) = \det(-P(\xi))\det(\lambda' - iQ\xi)$ had a non-zero real part. Since $\lambda'(-\xi,0) = -\lambda'(\xi',0)$ we may assume

$$\mathrm{Re}\ \lambda_1'(\xi_0,0) \geq c > 0 .$$

By restricting β to $0 \leq \beta \leq \epsilon$, we obtain

$$\mathrm{Re}\ \lambda_1'(\xi_0,\beta) \geq \frac{1}{2} c > 0 .$$

But then $\lambda_1(\beta^{-1}\xi_0) = \beta^{-1}\lambda_1'(\xi_0,\beta)$ has arbitrarily large real part as $\beta \to 0$. This shows that (3.1) is ill-posed if (3.5) has roots which are not imaginary.

<u>Theorem 3.2</u>. If (3.3) is Petrovskii parabolic and (3.4) is hyperbolic, then (3.1) is well-posed.

Proof. Our proof will rely on the theory of pseudo-differential operators. We define the symbols

$$P(x,t,\xi) = -\sum P_{ij}(x,t)\xi_i\xi_j \ ,$$

$$Q(x,t,\xi) = i \sum Q_k(x,t)\xi_k \ ,$$

and similarly $A(x,t,\xi)$ and $B(x,t,\xi)$. We set $H_2(x,t,\xi) = T^*(x,t,\xi)T(x,t,\xi)$. where $T(x,t,\xi)$ is as in Definition 2.2. Following Kreiss [7], we can construct a pseudo-differential operator $H_1(x,t,\xi)$ so that we have

$$(3.8)\quad \begin{cases} \text{a)} \quad \operatorname{Re} H_1 P \le (-\delta\xi^2 + \gamma)H_1 \\ \qquad \operatorname{Re} H_2 Q \le \gamma H_2 \\ \\ \text{b)} \quad H_1 \text{ and } H_2 \text{ have Hermitian symbols and} \\ \qquad c_0^{-1} \le H_1,\ H_2 \le c_0 \\ \qquad \text{for some positive constants } c_0,\ \gamma,\ \delta. \end{cases}$$

Now for a solution (u,v) to (3.1) we set

$$E = (u,H_1u) + (v,H_2v)$$

where the inner product is the usual L^2 inner product on $\mathbb{R}^{n+1}$. We then have

$$\begin{aligned} E_t &= 2\operatorname{Re}\{(u,H_1u_t) + (v,H_2v_t)\} + (u,H_{1t}u) + (v,H_{2t}v) \\ &= 2\operatorname{Re}\{(u,H_1Pu) + (v,H_2Qv) + (u,H_1Av) + (v,H_2Bu) \\ &\quad + (u,H_1F_1) + (v,H_2F_2)\} + (u,H_{1t}u) + (v,H_{2t}v). \end{aligned}$$

Now using Gårding's inequality and (3.8) we can estimate the first two terms by

$$CE - \delta(\nabla u, H_1 \nabla u),$$

and the second two terms by

$$C_\epsilon E + \epsilon(\nabla u, H_1 \nabla u),$$

and the last four terms by

$$CE + \|F\|^2 .$$

So we have

$$E_t \le CE + \|F\|^2 - c\|\nabla u\|^2 \tag{3.9}$$

and this implies (3.2).

We shall need two other energy inequalities that are proved in a similar way.

$$\|w(t)\|_r^2 + \int_0^t \|\nabla u(t')\|_r^2 \, dt' \le C_{T,r}(\|w_0\|_r^2 + \int_0^t \|F(t')\|_r^2 \, dt') \tag{3.10}$$

for all $t \in [0,T]$ and all integers r.

$$\|w\|_\eta^2 + \|\nabla u\|_\eta^2 \le C(\|w_0\|_0^2 - \mathrm{Re}(w, HF)_\eta) \tag{3.11}$$

where

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}.$$

In (3.10) the norms are those of the Sobolev space $H^r(\mathbb{E}^{n+1})$.

We now give a short proof for the existence of a solution for the Cauchy problem (3.1). We modify the proof given in Taylor [19] for symmetric hyperbolic systems.

For convenience we will write $w = (u,v)' \in H^{r_1,r_2}(\Omega)$ if $u \in H^{r_1}(\Omega)$ and $v \in H^{r_2}(\Omega)$. Let $H^r = H^r(\mathbb{R}^{n+1})$ and $H^{r,r} = H^r$.

<u>Theorem 3.3</u>. If $w_0 \in H^r$ and $F \in L^2([0,T],H^r)$ then the Cauchy problem (3.1) has a solution $w(t)$ such that $w \in L^2([0,T],H^{r+1,r})$ and $w \in C([0,T],H^{r,r})$.

<u>Proof</u>.

We abbreviate (3.1) by

$$w_t = K(x,t,D)w + F. \tag{3.12}$$

Let $J_\epsilon(x)$ be a Friedrichs mollifier on $\mathbb{R}^{n+1}$ for $0 < \epsilon \leq 1$, and consider the equation

$$\begin{aligned} w_t &= KJ_\epsilon w + F \\ &= K_\epsilon(x,t,D)w + F \end{aligned} \tag{3.13}$$

with $w = w_0$ at $t = 0$. Equation (3.13) is an ordinary differential equation on H^r. Using the standard Picard iteration procedure, we obtain a solution w^ϵ of (3.13) and $w^\epsilon \in C([0,T],H^r)$. Let

$$H(x,t,\xi) = \begin{pmatrix} H_1(x,t,\xi) & 0 \\ 0 & H_2(x,t,\xi) \end{pmatrix} .$$

It was shown in the proof of Theorem 3.2, that

$$H(x,t,\xi)\, K(x,t,\xi) + K^*(x,t,\xi)\, H(x,t,\xi)$$

$$\leq \begin{pmatrix} -c_1\xi^2 + c_0 & 0 \\ 0 & c_0 \end{pmatrix} H(x,t,\xi).$$

It then follows easily that

$$HK_\epsilon + K_\epsilon^* H \leq \begin{pmatrix} -c_1'\xi^2 + c_0' & 0 \\ 0 & c_0' \end{pmatrix} H$$

where the constants c_1' and c_0' are independent of ϵ. This implies,

$$(3.14) \qquad \|w^\epsilon(t)\|_r^2 + \int_0^t \|\nabla u^\epsilon(t')\|_r^2\, dt'$$

$$\leq C_{r,T}(\|w_0\|_r^2 + \int_0^t \|F(t')\|_r^2\, dt')$$

for all $\epsilon \in (0,1]$.

Equation (3.14) implies that $\{w^\epsilon\}$ is a bounded subset of $L^2([0,T],H^{r+1,r})$. It follows from (3.13) that $\{w_t^\epsilon\}$ is a bounded subset of $L^2([0,T],H^{r-1,r-1})$. Integrating we see that

(3.15) $\qquad \{w^\epsilon\}_{0<\epsilon\le 1}$ is an equicontinuous family in $C([0,T],H^{r-1,r-1})$.

Equation (3.14) also implies that

(3.16) $\qquad \{w^\epsilon(t)\}_{0<\epsilon\le 1}$ is a bounded subset of H^r for each $t \in [0,T]$.

We now apply Rellich's compactness theorem (Agmon [1], page 30ff) with Ascoli's theorem to obtain: 1) a subsequence $\{w^{\epsilon_k}\}_{k=1}^\infty$ of $\{w^\epsilon\}_{0<\epsilon\le 1}$ where $\epsilon_k \to 0$ as $k \to \infty$, and 2) a function $w \in C([0,T],H^{r-1})$ such that $w^{\epsilon_k}(t)$ converges to $w(t)$ in $H^{r-1}(\Omega)$ uniformly in t on each bounded open set $\Omega \subseteq \mathbb{R}^{n+1}$. Therefore $w(t)$ satisfies (3.12) weakly.

We have shown that we have a solution $w \in C([0,T],H^{r-1})$ for each $w_0 \in H^r$ and $F \in L^2([0,T],H^r)$. We now show that $w \in C([0,T],H^r)$. Let $w_{0,j} \in H^{r+1}$ and $F_j \in L^2([0,T],H^{r+1})$ converge to w_0 in H^r and F in $L^2([0,T],H^r)$ respectively. Then as above we obtain

$$w_j \in C([0,T],H^r)$$

which solves

$$w_t = Kw + F_j \quad \text{with} \quad w(0) = w_{0,j} .$$

Using the energy inequality (3.10) we see that w_j is a Cauchy sequence in $C([0,T],H^r)$ which is complete. It follows from uniqueness that w_j must converge to w. We see that $w \in C([0,T),H^r)$, and the energy inequality (3.10) shows $w \in L^2([0,T],H^{r+1,r})$.

CHAPTER IV

THE INITIAL BOUNDARY VALUE PROBLEM ON A HALF-SPACE

We will now consider the initial boundary value problem for the system (1.1) on a half-space. We begin by considering the case with constant coefficients and without lower order terms. Later we extend this to systems with variable coefficients and lower order terms.

4.1. Preliminaries

We rewrite (1.1) as

$$(4.1)\qquad u_t = P_0 u_{xx} + \sum_{k=1}^{n} P_{1k} u_{xy_k} + \sum_{j,k=1}^{n} P_{2jk} u_{y_j y_k} + A_0 v_x + \sum_{k=1}^{n} A_k v_{y_k} + F_1(x,y,t)$$

$$v_t = B_0 u_x + \sum_{k=1}^{n} B_k u_{y_k} + Q_0 v_x + \sum_{k=1}^{n} Q_k v_{y_k} + F_2(x,y,t)$$

on the region $x \geq 0$, $y \in \mathbb{R}^n$, $t \geq 0$. On the boundary $x = 0$, we impose the boundary conditions

$$(4.2)\qquad \begin{aligned} T_1 u_x + \sum_{k=1}^{n} T_{2k} u_{y_k} + S_1 v &= g_1(y,t) \\ Tu + Sv &= g_2(y,t) \end{aligned}$$

where g_1 and g_2 are vectors of dimensions b_1 and b_2, respectively.

We will also assume

$$u = 0 \quad \text{and} \quad v = 0 \quad \text{at} \quad t = 0 \,. \tag{4.3}$$

As in Chapter I, we will assume that the system

$$u_t = P_0 u_{xx} + \sum_{k=1}^{n} P_{1k} u_{xy_k} + \sum_{j,k=1}^{n} P_{2jk} u_{y_j y_k}$$

is Petrovskii parabolic and we make the further assumption that the system

$$v_t = Q_0 v_x + \sum_{k=1}^{n} Q_k v_{y_k} \tag{4.4}$$

is strictly hyperbolic.

Recall that u has dimension p and v has dimension q.

Without loss of generality we may assume that Q_0 is diagonal and

$$Q_0 = \begin{bmatrix} Q_0^- & 0 \\ 0 & Q_0^+ \end{bmatrix}, \qquad Q_0^- < 0, \; Q_0^+ > 0 \,, \tag{4.5}$$

where Q_0^- and Q_0^+ are diagonal matrices of dimension $q^- \times q^-$ and $q^+ \times q^+$, respectively. In particular, note that we are assuming that the boundary is non-characteristic.

We begin to analyze the system (4.1) by Fourier transforming in the y variables and Laplace transforming in the t variable. Set

$$\hat{w}(x,\omega,s) = (2\pi)^{-(n+1)/2} \int_{\mathbb{R}^n} \int_0^\infty e^{-st} e^{-i\omega y} w(x,y,t)\, dy\, dt$$

This transforms (4.1) to an ordinary differential equation with the independent variable x. In what follows we shall not write the "^" on the transformed dependent variables, this abuse of notation should cause no difficulty.

We can express the resulting ordinary differential equation as a first order system by defining an additional dependent variable

$$\tilde{u} = (u_x + P_0^{-1}A_0 v)(\omega^2 + s)^{-1/2} ,$$

also let $\sigma = (\omega^2 + s)^{1/2}$. We then have

$$(4.6)\quad \begin{pmatrix} u \\ \tilde{u} \\ v \end{pmatrix}_x = \begin{pmatrix} 0 & \sigma & 0 \\ P_0^{-1}(s - P_2(\omega))\sigma^{-1} & -iP_0^{-1}P_1\cdot\omega & 0 \\ -iQ_0^{-1}B_2\cdot\omega & -Q_0^{-1}B_0\sigma & M(\omega,s) \end{pmatrix} \begin{pmatrix} u \\ \tilde{u} \\ v \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & -P_0^{-1}A_0 \\ 0 & 0 & (iP_0^{-1}P_1\cdot\omega P_0^{-1}A_0 + iA\cdot\omega)\sigma^{-1} \\ 0 & 0 & Q_0^{-1}B_0P_0^{-1}A_0 \end{pmatrix} \begin{pmatrix} u \\ \tilde{u} \\ v \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ -P_0^{-1}F_1\sigma^{-1} \\ -Q_0^{-1}F_2 \end{pmatrix} ,$$

where $$M_2(\omega,s) = Q_0^{-1}(s - iQ\cdot\omega),$$

$$P_2(\omega) = \sum_{j,k=1}^{n} P_{2jk}\omega_j\omega_k$$

The boundary conditions at $x = 0$ become

$$(4.7) \qquad \begin{pmatrix} iT_2\cdot\omega\,\sigma^{-1} & T_1 & 0 \\ T & 0 & S \end{pmatrix}\begin{pmatrix} u \\ \tilde{u} \\ v \end{pmatrix}$$

$$= \begin{pmatrix} g_1\sigma^{-1} \\ g_2 \end{pmatrix} + \begin{pmatrix} (T_1P_0^{-1}A - S_1)\sigma^{-1}v \\ 0 \end{pmatrix}.$$

In both (4.6) and (4.7) the second term on the right-hand side is a lower order term and does not affect the following analysis.

Letting $w = (u,\tilde{u},v)'$ we can rewrite (4.6) and (4.7) as

$$(4.8) \qquad \begin{aligned} w_x &= N(\omega,s)w + F(x,\omega,s) \\ T'(w,s)w &= g(\omega,s) \quad \text{at } x = 0 \;, \end{aligned}$$

where we have dropped the lower order terms, and $F = (0,-P_0^{-1}F_1\sigma^{-1},-Q_0^{-1}F_2)'$, $g = (g_1\sigma^{-1},g_2)'$.

We now pause in our analysis to consider what we mean by "well-posed". Perhaps the most natural definition for the well-posedness of the system (4.1-4.3) is:

Definition 4.2. The initial boundary value problem (4.1-4.3) is well-posed if there are constants η_0 and C_0 such that for any solution we have

$$\|u\|_\eta^2 + \|v\|_\eta^2 + |u|_\eta^2 + |v|_\eta^2 \leq C_0(|g_1|_\eta^2 + |g_2|_\eta^2 + \|F_1\|_\eta^2 + \|F_2\|_\eta^2) \tag{4.9}$$

for all $\eta \geq \eta_0$. (Recall $\|\cdot\|_\eta = \|\cdot\|_{\eta,\Omega}$ and $|\cdot|_\eta = |\cdot|_{\eta,\partial\Omega}$, $\Omega = \mathbb{R}_+^{n+1}$.)

However, this definition is inadequate for our analysis via equation (4.8). It will become apparent later that the natural definition of "well-posed" for the system (4.8) is the following.

Definition 4.3. The system (4.8) is σ-well-posed if there are constants η_0 and C_0 such that for any $L^2(\mathbb{E}_+)$ solution of (4.8) with $\eta > \eta_0$, we have

$$\operatorname{Re}\sigma(\|u\|_+^2 + \|\tilde{u}\|_+^2) + \eta\|v\|_+^2 + |u|^2 + |\tilde{u}|^2 + |v|^2 \leq C_0(|g|^2 + \|F\|_+^2) \quad . \tag{4.10}$$

This then gives us an alternate definition for the well-posedness of (4.1-4.3). First, let Σ and Σ^{-1} be the pseudo-differential operators with symbols $\sigma = (\omega^2 + s)^{1/2}$ and $\sigma^{-1} = (\omega^2 + s)^{-1/2}$, respectively. (We will give a short discussion of pseudo-differential operators that will be sufficient for our needs in Chapter VII.) In analogy to Definition 4.3 we make the following definition.

Definition 4.4. The initial boundary value problem (4.1-4.3) is σ-well-posed if there are constants η_0 and C_0 such that for any solution we have

$$(4.11) \qquad \mathrm{Re}(u,\Sigma u)_\eta + \mathrm{Re}(u_x,\Sigma^{-1}u_x)_\eta + \eta\|v\|_\eta^2 + |u|_\eta^2 + |\Sigma^{-1}u_x|_\eta^2 + |v|_\eta^2$$
$$\leq C_0(|\Sigma^{-1}g_1|_\eta^2 + |g_2|_\eta^2 + \|\Sigma^{-1}F_1\|_\eta^2 + \|F_2\|_\eta^2)$$

for all $\eta \geq \eta_0$.

The problem will be said to be σ-ill-posed if it is not σ-well-posed.

We then have the following two theorems.

Theorem 4.1. The initial boundary value problem (4.1-4.3) is σ-well-posed if and only if the system (4.8) is σ-well-posed.

Theorem 4.2. If the initial boundary value problem (4.1-4.3) is σ-well-posed then it is well-posed.

We will give the proof of Theorem 4.1 later.

Proof of Theorem 4.2.

Equation (4.9) follows from (4.11) since

$$\sqrt{\eta}\ \|u\|_\eta \le \operatorname{Re}(u,\Sigma u)_\eta , \qquad 0 \le \operatorname{Re}(u_x,\Sigma^{-1}u_x)_\eta,$$

and

$$\|\Sigma^{-1}g\|_\eta \le \eta^{-1/2}\ \|g\|_\eta .$$

4.2. The matrix $N(\omega,s)$

It is essential for the following analysis that we examine the matrix $N(\omega,s)$ closely. We write

$$N(\omega,s) = \begin{pmatrix} N_{11}(\omega,s) & 0 \\ N_{21}(\omega,s) & N_{22}(\omega,s) \end{pmatrix} \tag{4.12}$$

where

$$N_{11}(\omega,s) = \begin{pmatrix} 0 & \sigma \\ P_0^{-1}\dfrac{s - P_2(\omega)}{\sigma} & -iP_0^{-1}P_1\cdot\omega \end{pmatrix},$$

$$N_{21}(\omega,s) = (-iQ_0^{-1}B\cdot\omega, -Q_0^{-1}B_0\sigma) ,$$

and

$$N_{22}(\omega,s) = M(\omega,s) = Q_0^{-1}(s - iQ\cdot\omega) .$$

N_{11} is a $2p \times 2p$ matrix and N_{22} is a $q \times q$ matrix, and so the eigenvalues of $N(\omega,s)$ are precisely those of N_{11} and N_{22}. We also note the homogeneity properties of the components of N. N_{11} and N_{21} have parabolic homogeneity and N_{22} has hyperbolic homogeneity, i.e. for $\rho > 0$

$$
\begin{aligned}
&(4.13) \qquad N_{11}(\rho\omega,\rho^2 s) = \rho N_{11}(\omega,s)\ , \\
&\qquad\qquad N_{21}(\rho\omega,\rho^2 s) = \rho N_{21}(\omega,s)\ , \\
&\text{and} \\
&\qquad\qquad N_{22}(\rho\omega,\rho s) = \rho N_{22}(\omega,s)\ .
\end{aligned}
$$

We now examine the eigenvalues of N.

<u>Theorem 4.3.</u> The eigenvalues λ of $N_{11}(\omega,s)$ are the roots of

$$(4.14) \qquad \det(P_0\lambda^2 + iP_1\cdot\omega\lambda + P_2(\omega) - s) = 0\ .$$

For $\operatorname{Re} s \geq -\alpha_1\omega^2$, $(\omega,s) \neq 0$, there are p eigenvalues with positive real parts and p with negative real parts. (α_1 is a positive constant less than α, (Definition 2.1).) Moreover, for $\operatorname{Re} s \geq -\alpha_1\omega^2$ there is a positive constant c such that

$$(4.15) \qquad |\operatorname{Re}\lambda| \geq c\ \sqrt{\omega^2 + |s|}.$$

Proof. Equation (4.14) follows easily from the definition of N_{11}. For $\operatorname{Re} s \geq -\alpha_1\omega^2$, $(\omega,s) \neq 0$, if λ were pure imaginary we would have, by Definition 2.1,

$$\operatorname{Re} s \leq -\alpha(|\lambda|^2 + \omega^2) < -\alpha_1\omega^2 \leq \operatorname{Re} s$$

which is a contradiction. So $\operatorname{Re}\lambda(\omega,s)$ is not zero. For $\omega = 0$, $\operatorname{Re} s \geq 0$, $\lambda(0,s)$ satisfies

$$\det(P_0\lambda^2 - s) = 0,$$

which shows that the λ's are the eigenvalues of $\pm\sqrt{s}\,P_0^{-1/2}$ and hence split into two groups of p elements each. One group contains the eigenvalues with positive real parts, and the other those with negative real parts.

Finally for $\omega^2 + |s| = 1$, $\operatorname{Re} s \geq -\alpha_1\omega^2$, we have $|\operatorname{Re}\lambda| \geq c > 0$ by compactness. Then (4.15) follows by homogeneity.

We have an analogous theorem for N_{22}.

Theorem 4.4. For $\operatorname{Re} s > 0$, the eigenvalues of $N_{22}(\omega,s)$ split into two groups. There are q^- eigenvalues μ_- with $\operatorname{Re}\mu_- < 0$ and q^+ eigenvalues μ_+ with $\operatorname{Re}\mu_+ > 0$.

Proof. If μ is an eigenvalue of $N_{22}(\omega,s)$,

$$0 = \det(\mu - N_{22}(\omega,s)) = \det Q_0^{-1} \det(s - Q_0\mu - iQ\cdot\omega).$$

So if μ is pure imaginary, s must also be pure imaginary, hence $\operatorname{Re} s \neq 0$ implies $\operatorname{Re}\mu \neq 0$. For $\operatorname{Re} s > 0$ and $\omega = 0$, μ is an eigenvalue of sQ_0^{-1} and by (4.5) the theorem follows easily.

4.3. The Boundary Conditions

From the above we see that the space of $L^2(\mathbb{R}_+)$ solutions of the differential equation (4.8) has dimension $p + q^-$ when $\mathrm{Re}\ s > 0$. So, it is necessary to give at least $p + q^-$ boundary conditions to insure uniqueness of the solution. If more than $p + q^-$ boundary conditions are given we cannot insure existence. Because of this we make the following assumption.

Assumption 4.1. We assume that there are precisely $p + q^-$ boundary conditions, i.e.

$$b_1 + b_2 = p + q^- .$$

Moreover, we assume that b_1 is minimal in the sense that any linear combination of the rows of (4.2) does not diminish the number of rows containing derivatives.

The assumption on the minimality of b_1 is inserted since adding one of the first b_1 rows to all the rows would make a σ-well-posed problem become σ-ill-posed. This would be tantamount to replacing the term $|g_2|_\eta^2$ in (4.11) by the stronger $|\Sigma^{-1}g_2|_\eta^2$.

We now define the eigensolutions of the system (4.8) and of (4.1-4.3).

Given any vector v of dimension q, we can decompose it as $v = (v^-, v^+)'$ where v^- consists of the first q^- components and v^+ consists of the last q^+ components.

Definition 4.5. (u,v_0) is an eigensolution of parabolic type at (ω,s) if it satisfies:

(4.16) a) $(u,v_0) \neq 0$,

b) $(\omega,s) \neq 0$, $\mathrm{Re}\, s \geq 0$,

c) $su = P_0 u_{xx} + iP_1 \cdot \omega u_x + P_2(\omega)u$,

d) $v_0^+ = 0$,

e) $T_1 u_x + iT_2 \cdot \omega u = 0$

$Tu + Sv_0 = 0$ at $x = 0$,

f) $u \in L^2(\mathbb{R}_+)$ and $v_0 \in \mathbb{C}^q$.

Definition 4.6. (u,v) is an eigensolution of hyperbolic type at (ω,s) if it satisfies

(4.17) a) $(u,v) \neq 0$,

b) $\omega \neq 0$, $\mathrm{Re}\, s \geq 0$,

c) $0 = P_0 u_{xx} + iP_1 \cdot \omega u + P_2(\omega)u$,

d) $sv = B_0 u_x + iB \cdot \omega u + Q_0 v_x + iQ \cdot \omega v$,

e) $T_1 u_x + iT_2 \cdot \omega u = 0$

$Tu + Sv = 0$ at $x = 0$,

f) For $\mathrm{Re}\, s > 0$, u and v are in $L^2(\mathbb{R}_+)$, and for $\mathrm{Re}\, s = 0$, u is $L^2(\mathbb{R}_+)$ and v is the limit as $\mathrm{Re}\, s \to 0^+$ of $L^2(\mathbb{R}_+)$ solutions to d).

We now state the main theorem of this chapter.

Theorem 4.5. The initial boundary value problem (4.1-4.3) is σ-well-posed if and only if it has no eigensolutions of either parabolic or hyperbolic type.

Theorem 4.5 follows immediately from Theorem 4.1 and the following theorem.

Theorem 4.6. The system (4.8) is σ-well-posed if and only if it has no eigensolutions of either parabolic or hyperbolic type.

4.4. The Proof of Theorem 4.6

Before we give the proof of Theorem 4.6 we introduce some notation. By Theorems 4.3 and 4.4 we have for $\mathrm{Re}\ s > 0$

$$\mathbb{C}^{2p+q} = E^-(\omega,s) \oplus E^+(\omega,s)$$

where $E^-(\omega,s)$ (resp. $E^+(\omega,s)$) is the span of the generalized eigenvectors of $N(\omega,s)$ whose eigenvalues have negative (resp. positive) real parts. For $w \in \mathbb{C}^{2p+q}$ we write

$$w = w^-(\omega,s) + w^+(\omega,s)$$

where

$$w^{\pm}(\omega,s) \in E^{\pm}(\omega,s)$$

The proof of Theorem 4.6 depends heavily on the following theorem whose proof is given in Chapter VII.

<u>Theorem 4.7</u>. For the system (4.8) there exists a Hermitian matrix $R(\omega,s)$ satisfying the following:

(4.18) a) $\operatorname{Re} R(\omega,s)\, N(\omega,s) \geq d(\omega,s)$ where

$$d(\omega,s) = \begin{pmatrix} \sigma_0 I_{2p} & 0 \\ 0 & \eta I_q \end{pmatrix},$$

$$\sigma_0 = \operatorname{Re} \sigma = \operatorname{Re}(\omega^2 + s)^{1/2}.$$

b) $w^t \; R(\omega, s) w \geq c_0(\delta |w^+(\omega,s)|^2 - |w^-(\omega,s)|^2)$,

where δ is a positive constant which may be taken arbitrarily large, $c_0 > 0$.

c) $R(\omega,s)$ is a C^∞ function of (ω,s) for $\omega \in \mathbb{R}^n$, $\operatorname{Re} s \geq 0$.

d) The norm of $R(\omega,s)$ is bounded independently of (ω,s).

e) The lower left $q \times 2p$ submatrix of Re RN is zero for $|s| \geq c_1|\omega|$, $|s| \geq c_2$.

<u>The proof of Theorem 4.6</u>.

We begin by showing that the nonexistence of eigensolutions implies that (4.8) is well-posed. Then we show that the existence of eigensolutions implies that (4.8) is σ-ill-posed.

Applying the results of Theorem 4.7 we have

(4.19)
$$\begin{aligned} \sigma_0(\|u\|_+^2 + \|\tilde{u}\|_+^2) + \eta\|v\|_+^2 &\leq \operatorname{Re}(w,RNw)_+ \\ &= \operatorname{Re}(w,Rw_x)_+ - \operatorname{Re}(w,RF)_+ \\ &= -\frac{1}{2} w^t Rw_{(x=0)} - \operatorname{Re}(w,RF)_+ \\ &\leq \frac{1}{2} c_0(|w^-|^2 - \delta|w^+|^2) + \|w\|_+^2 + C\|F\|_+^2 \,. \end{aligned}$$

We now write the boundary condition of (4.8) as

$$T'(\omega,s)w^- = g(\omega,s) - T'(\omega,s)w^+,$$

and show that we have the estimate

$$|w^-| \le C_0|T'(\omega,s)w^-| \tag{4.20}$$

for some constant C_0 and $\operatorname{Re} s = \eta \ge \eta_0$ for some η_0.

Suppose that (4.20) fails, then there is a sequence

$$\{(\omega_\nu,s_\nu)\}_{\nu=1}^{\infty} \quad \text{with} \quad \operatorname{Re} s_\nu \to \infty , \tag{4.21}$$

and solutions $w_\nu \in L^2(\mathbb{R}_+)$ to

$$w_{\nu,x} = N(\omega_\nu,s_\nu)w_\nu$$

with $|w_\nu(x = 0)| = 1$ and $|T'(\omega_\nu,s_\nu)w_\nu| \to 0$. There are two cases to consider. First suppose that for the sequence (4.21),

$$|\omega_\nu|/|s_\nu| \ge a > 0 \tag{4.22}$$

for some constant a. In this case we normalize as follows:

$$\rho_\nu = (|\omega_\nu|^2 + |s_\nu|^2)^{1/2} ,$$

$$s'_\nu = s_\nu/\rho_\nu, \qquad \omega'_\nu = \omega_\nu/\rho_\nu, \qquad x' = x\rho_\nu ,$$

$$w_\nu(x',\omega'_\nu,s'_\nu) = w_\nu(x,\omega_\nu,s_\nu) .$$

Now since $|\omega'_\nu|^2 + |s'_\nu|^2 = 1$ and $|w_\nu(0)| = 1$ there is a subsequence such that (ω'_ν, s'_ν) and $w_\nu(0)$ converge to (ω', s') and w_0. Without loss of generality we can assume that the original sequence converges. In terms of the normalized quantities we have

$$w_{\nu,x'} = \frac{1}{\rho_\nu} w_{\nu,x} = \frac{1}{\rho_\nu} N(\omega_\nu, s_\nu) w_\nu = \frac{1}{\rho_\nu} N(\rho_\nu \omega'_\nu, \rho_\nu s'_\nu) w_\nu \ .$$

We now consider the components of $N(\omega, s)$.

$$\frac{1}{\rho_\nu} N_{11}(\rho_\nu \omega'_\nu, \rho_\nu s'_\nu) = N_{11}(\omega'_\nu, \rho_\nu^{-1} s'_\nu) \to N_{11}(\omega', 0)$$

and similarly for N_{21}, and

$$\frac{1}{\rho_\nu} N_{22}(\rho_\nu \omega'_\nu, \rho_\nu s'_\nu) = N_{22}(\omega'_\nu, s'_\nu) \to N_{22}(\omega', s') \ .$$

So that in the limit

$$w_{0,x} = \begin{pmatrix} N_{11}(\omega', 0) & 0 \\ N_{21}(\omega', 0) & N_{22}(\omega', s') \end{pmatrix} w_0 \quad ,$$

and the boundary condition becomes

$$\begin{pmatrix} iT_2 \cdot \dfrac{\omega'}{|\omega'|} & T_1 & 0 \\ T & 0 & S \end{pmatrix} w_0 = 0 \ .$$

Also note that $|\omega'| \geq a/\sqrt{1+a^2} > 0$. But these are precisely the conditions for (u,v) to be an eigensolution of hyperbolic type at (ω',s') where $w_0 = (u,\tilde{u},v)$.

The second case to consider is when

$$\frac{|\omega_\nu|}{|s_\nu|} \to 0 \quad \text{as} \quad \nu \to \infty \tag{4.23}$$

We normalize as follows

$$\rho_\nu = (|\omega_\nu|^2 + |s_\nu|)^{1/2} ,$$

$$s'_\nu = s_\nu/\rho^2_\nu, \qquad \omega'_\nu = \omega_\nu/\rho_\nu, \qquad x' = x\rho_\nu, \qquad x'' = x|s_\nu|.$$

Again, we can assume that (ω'_ν, s'_ν) and $w_\nu(0)$ converge to (ω',s') and w_0. We can also assume that $s_\nu/|s_\nu|$ converges to s''. We then have

$$\begin{pmatrix} u_\nu \\ \tilde{u}_\nu \end{pmatrix}_{x'} = N_{11}(\omega'_\nu, s'_\nu) \begin{pmatrix} u_\nu \\ \tilde{u}_\nu \end{pmatrix}$$

and

$$v_{\nu,x''} = N_{22}\left(\frac{\omega_\nu}{|s_\nu|}, \frac{s_\nu}{|s_\nu|}\right) v_\nu + \frac{\rho_\nu}{|s_\nu|} N_{21}(\omega'_\nu, s'_\nu) \begin{pmatrix} u_\nu \\ \tilde{u}_\nu \end{pmatrix} .$$

Now

$$\frac{\rho_\nu}{|s_\nu|} = \left(\frac{1}{|s_\nu|} + \left|\frac{\omega_\nu}{s_\nu}\right|^2\right)^{1/2} \to 0$$

and $(u_\nu, \tilde{u}_\nu)'$ is a bounded function, so in the limit $v(x'')$ satisfies

$$v_{x''} = N_{22}(0,s'')v = Q_0^{-1}s''v \ .$$

But this implies $v_0^+ = 0$. We also have

$$\begin{pmatrix} u_0 \\ \tilde{u}_0 \end{pmatrix}_{x'} = N_{11}(\omega',s') \begin{pmatrix} u_0 \\ \tilde{u}_0 \end{pmatrix}$$

and the boundary condition becomes

$$T'(\omega',s')w_0 = 0 \quad .$$

These conditions are precisely those for $(u_0(x'),v_0)$ to be an eigensolution of parabolic type at (ω',s').

Now any sequence (4.21) has a subsequence satisfying either (4.22) or (4.23) and hence generates an eigensolution of either hyperbolic or parabolic type. So from the non-existence of eigensolutions an estimate of the form (4.20) holds for some constants C_0 and η_0. Applying (4.20) to (4.19) we have

$$\begin{aligned}
&\sigma_0(\|u\|_+^2 + \|\tilde{u}\|_+^2) + \eta\|v\|_+^2 \\
&\quad \leq 2C_0|g - T'(\omega,s)w^+|^2 - (|w^-|^2 + \delta|w^+|^2) + \|w\|_+^2 + C\|F\|_+^2 \\
&\quad \leq C(|g|^2 + \|F\|_+^2) - |w^-|^2 - |w^+|^2\ (\delta - C_1) + \|w\|_+^2 \ .
\end{aligned}$$

Now for η sufficiently large and δ taken appropriately, we obtain

$$\sigma_0(\|u\|_+^2 + \|\tilde{u}\|_+^2) + \eta\|v\|_+^2 + |w|^2 \le C(|g|^2 + \|F\|_+^2) \quad .$$

This is equivalent to (4.10). So if there are no eigensolutions then the problem is σ-well-posed.

We now show that the existence of eigensolutions implies that (4.8) is σ-ill-posed.

Definition 4.7. For any matrix X let $E^-(X)$ be the span of the generalized eigenvectors of X whose corresponding eigenvalues have negative real part. Similarly for $E^+(X)$.

Assume that (u,v) is an eigensolution of hyperbolic type at (ω_0, s_0). Let

$$w(x) = \begin{pmatrix} u(x) \\ u_x(x)/|\omega_0| \\ v(x) \end{pmatrix} .$$

Then w satisfies

$$w_x = \begin{pmatrix} N_{11}(\omega_0, 0) & 0 \\ N_{21}(\omega_0, 0) & N_{22}(\omega_0, s_0) \end{pmatrix} w \tag{4.24}$$

with the boundary condition

$$\begin{pmatrix} iT_2 \cdot \frac{\omega_0}{|\omega_0|} & T_1 & 0 \\ T & 0 & S \end{pmatrix} w(0) = 0 .$$

Call the matrix in (4.24) $N^*(\omega_0, s_0)$. From the definition of an eigensolution of the hyperbolic type we have

$$w(0) \in E^-(N^*(\omega_0, s_0)) \qquad \text{if} \quad \operatorname{Re} s_0 > 0 ,$$

and

$$w(0) \in \lim_{\epsilon \to 0^+} E^-(N^*(\omega_0, s_0 + \epsilon)) \qquad \text{if} \quad \operatorname{Re} s_0 = 0 .$$

We can assume $|w(0)| = 1$.

Now consider the equation

$$w_x^\rho = N(\rho\omega_0, \rho s_0 + \bar\eta) w$$

$$= \begin{pmatrix} N_{11}(\rho\omega_0, \rho s_0 + \bar\eta) & 0 \\ N_{21}(\rho\omega_0, \rho s_0 + \bar\eta) & N_{22}(\rho\omega_0, \rho s_0 + \bar\eta) \end{pmatrix} w,$$

where $\bar\eta = 0$ if $\operatorname{Re} s_0 > 0$ and is chosen positive otherwise.

Now

$$E^-(N(\rho\omega_0, \rho s_0 + \bar\eta)) = E^-(\tfrac{1}{\rho} N(\rho\omega_0, \rho s_0 + \bar\eta))$$

and

$$\frac{1}{\rho} N(\rho\omega_0, \rho s_0 + \eta) = \begin{pmatrix} N_{11}\left(\omega_0, \frac{s_0}{\rho} + \frac{\bar{\eta}}{\rho^2}\right) & 0 \\ N_{21}\left(\omega_0, \frac{s_0}{\rho} + \frac{\bar{\eta}}{\rho^2}\right) & N_{22}\left(\omega_0, s_0 + \frac{\bar{\eta}}{\rho}\right) \end{pmatrix}$$

So $\frac{1}{\rho} N(\rho\omega_0, \rho s_0 + \eta)$ is an analytic perturbation of $N^*(\omega_0, s_0)$. Following Kato [5], we can choose

$$w^\rho(0) \in E^-(N(\rho\omega_0, \rho s_0 + \eta))$$

such that

$$w^\rho(0) \to w(0) \quad \text{as} \quad \rho \to \infty .$$

Moreover, we have

$$g^\rho = T'(\rho\omega_0, \rho\omega_0 + \bar{\eta})w^\rho \to 0 \quad \text{as} \quad \rho \to \infty$$

Thus we have solutions w^ρ to (4.8) at (ω, s) with Re s arbitrarily large such that $F = 0$ and as $\rho \to \infty$, $|w^\rho(0)| \to 1$ and $|g^\rho| \to 0$. This shows (4.8) to be σ-ill-posed.

Now assume that (u, v_0) is an eigensolution of parabolic type at (ω_0, s_0). Let

$$\bar{u} = (u, u_x/\sigma) \quad \text{where} \quad \sigma = (\omega_0^2 + s_0)^{1/2} ,$$

then $\bar{u}$ satisfies.

$$\bar{u}_x = N_{11}(\omega_0,s_0)\bar{u}$$

and

$$\bar{u}(0) \in E^-(N_{11}(\omega_0,s_0)).$$

First assume $s_0 \neq 0$. Let $\omega = \rho\omega_0$, $s = \rho^2 s_0 + \bar{\eta}$ where $\bar{\eta}$ is as before. Consider the solutions to

$$w_x^\rho = N(\omega,s)w^\rho$$

with

$$w^\rho(0) \in E^-(N(\omega,s)) .$$

Since

$$\frac{1}{\rho} N(\omega,s) = \begin{pmatrix} N_{11}(\omega_0,\ s_0 + \bar{\eta}/\rho^2) & 0 \\ N_{21}(\omega_0,\ s_0 + \bar{\eta}/\rho^2) & \rho N_{22}(\omega_0/\rho,\ s_0 + \bar{\eta}/\rho^{-2}) \end{pmatrix} ,$$

we see that

$$E^-(N(\omega,s)) \to E^-(N_{11}(\omega_0,s_0)) \oplus E^-(N_{22}(0,s_0))$$

as $\rho \to \infty$. So we can choose $w^\rho(0)$ such that $w^\rho(0) \to (\bar{u}(0),v_0)$ as $\rho \to \infty$.

The boundary condition then satisfies

$$T'(\omega,s)\ w^\rho(0) = g^\rho \to 0 \qquad \text{as} \qquad \rho \to \infty$$

In the case $s_0 = 0$, let $s = \rho^{3/2}\ \bar{\eta}$ and we obtain a similar result. This implies that (4.8) is σ-ill-posed.

What we have now shown is that (4.8) is σ-ill-posed if and only if there are no eigensolutions of either hyperbolic or parabolic type. This proves Theorem 4.6.

4.5. The Proof of Theorem 4.1

Assume that (4.8) is σ-well-posed and let (u,v) be a solution to (4.1-4.3). Then their transforms satisfy (4.6), so we have

$$\operatorname{Re} \sigma(\|\hat{u}\|_+^2 + \|\tilde{u}\|_+^2) + \eta\|\hat{v}\|_+^2 + |\hat{u}|^2 + |\tilde{u}|^2 + |\hat{v}|^2$$

$$\leq C_0(|\sigma^{-1}(\hat{g}_1 + T_1 P_0^{-1} A_0 \hat{v})|^2 + |\hat{g}_2|^2 + \|\hat{F} + N_1(\omega,s)\hat{v}\|_+^2) ,$$

where $N_1(\omega,s)\hat{v}$ is the second term on the right-hand side of (4.6). Since $\tilde{u} = (\hat{u}_x + P_0^{-1}A_0\hat{v})\sigma^{-1}$ and $\|N_1(\omega,s)\| \leq C$, we have

$$\operatorname{Re} \sigma(\|\hat{u}\|_+^2 + \|\sigma^{-1}\hat{u}_x\|_+^2 - C\|\sigma^{-1}\hat{v}\|_+^2) + \eta\|\hat{v}\|_+^2 + |\hat{u}|^2 + |\sigma^{-1}\hat{u}_x|^2 - C|\sigma^{-1}\hat{v}|^2 + |\hat{v}|^2$$

$$\leq C_0(|\sigma^{-1}g_1|^2 + |\sigma^{-1}\hat{v}|^2 + |\hat{g}_2|^2 + \|\sigma^{-1}\hat{F}_1\|_+^2 + \|\hat{F}_2\|_+^2 + C\|\hat{v}\|_+^2) .$$

Now $|\sigma|^{-1} \leq \eta^{-1/2}$ so we have, for η large enough, that

$$\operatorname{Re} \sigma\|\hat{u}\|_+^2 + \operatorname{Re} \sigma^{-1}\|\hat{u}_x\|_+^2 + \eta\|\hat{v}\|_+^2 + |\hat{u}|^2 + |\sigma^{-1}\hat{u}_x|^2 + |\hat{v}|^2$$

$$\leq C_0'(|\sigma^{-1}\hat{g}_1|^2 + |\hat{g}_2|^2 + \|\sigma^{-1}\hat{F}_1\|_+^2 + \|\hat{F}_2\|_+^2) .$$

By Parseval's relation we have (4.11) which shows (4.1-4.3) is σ-well-posed.

Now suppose (4.8) is σ-ill-posed. Then as shown in the proof of Theorem 4.6, for any $\delta > 0$ there is a solution w to (4.8) at (ω_0, s_0) with

$$|w(0)| = 1, \qquad F = 0 , \tag{4.25}$$
$$\operatorname{Re} s_0 > \varepsilon^{-1}, \qquad |g| < \delta.$$

By considering the addition of the lower order terms in (4.6) we can obtain $L^2(\mathbb{R}_+)$ solutions to

$$s_0\bar{u} = P_0\bar{u}_{xx} + iP_1\cdot\omega_0\bar{u}_x + P_2(\omega_0)\bar{u} + A_0\bar{v}_x + iA\cdot\omega_0\bar{v}$$

$$s_0\bar{v} = B_0\bar{u}_x + iB\cdot\omega_0\bar{u} + Q_0\bar{v}_x + iQ\cdot\omega_0\bar{v} \quad ,$$

with the boundary conditions

$$\begin{aligned} T_1\bar{u}_x + iT_2\cdot\omega_0\bar{u} + S_1\bar{v} &= \bar{g}_1 \\ T\bar{u} \quad + \qquad\qquad S\bar{v} &= \bar{g}_2 \quad . \end{aligned}$$

Equation (4.25) is satisfied with

$$|g|^2 = |\sigma^{-1}\bar{g}_1|^2 + |\bar{g}_2|^2 < \delta^2 \ .$$

We now construct a solution to (4.1-4.3) as follows. Choose $\epsilon > 0$ and $\varphi(y)$ satisfying

$$\|\varphi\| = 1 \ ,$$

and

$$\hat{\varphi}(\omega) = 0 \qquad \text{if} \qquad |\omega - \omega_0| \geq \epsilon \ .$$

Let

$$u(x,y,t) = \bar{u}(x)\ \varphi(y)\ (e^{s_0t} - 1)$$

$$v(x,y,t) = \bar{v}(x)\ \varphi(y)\ (e^{s_0t} - 1) \ .$$

Then (u,v) satisfies (4.2) and (4.3) with

$$g_1 = \bar{g}_1 \varphi(y) (e^{s_0 t} - 1) + g_1'$$
$$g_2 = \bar{g}_2 \varphi(y) (e^{s_0 t} - 1)$$

where

$$g_1' = \int e^{i\omega y} \hat{\varphi}(\omega) i(\omega - \omega_0)\cdot T_2 \, d\omega \, \bar{u}(0) (e^{s_0 t} - 1) .$$

Also (u,v) satisfies (4.1) with

$$\hat{F}_1(x,\omega,t) = \hat{\varphi}(\omega)[iP_1\cdot(\omega_0-\omega) \bar{u}_x(x) + (P_2(\omega_0)-P_2(\omega))\bar{u}(x) + iA\cdot(\omega_0-\omega)\bar{v}(x)](e^{s_0 t}-1)$$
$$+ s_0\bar{u}(x) \hat{\varphi}(\omega)$$
$$\hat{F}_2(x,\omega,t) = \hat{\varphi}(\omega)[iB\cdot(\omega_0-\omega)\bar{u}(x) + iQ\cdot(\omega_0-\omega)\bar{v}(x)](e^{s_0 t}-1) + s_0\bar{v}(x) \hat{\varphi}(\omega) .$$

Now choose $\eta > \eta_0 = \operatorname{Re} s_0$ and set

$$q(\eta) = \int_0^\infty e^{-2\eta t}|e^{s_0 t} - 1|^2 \, dt .$$

Then

$$|u|_\eta^2 + |\Sigma^{-1}u_x|_\eta^2 + |v|_\eta^2 \geq q(\eta) (1 - C\epsilon^2) - \frac{c|s_0|}{\eta^{1/2}} ,$$

and we have the following estimates, here $\sigma_0 = (\omega_0^2 + s_0)^{1/2}$,

$$|\Sigma^{-1}g_1|_\eta^2 \leq q(\eta)[|\sigma_0^{-1} \bar{g}_1|^2 + \epsilon^2|\bar{u}(0)|^2] + \frac{c|s_0|}{\eta^{1/2}} ,$$

$$|g_2|_\eta^2 \leq q(\eta) |\bar{g}_2|^2 ,$$

$$\|\Sigma^{-1}F_1\|_\eta^2 \leq q(\eta)C\epsilon^2(|\sigma_0^{-1}\bar{u}_x|_+^2 + |\bar{u}|_+^2 + |\bar{v}|_+^2) + |s_0|^2 |\bar{u}|_+^2 \eta^{-1} ,$$

and

$$\|F_2\|_\eta^2 \le C\epsilon^2 q(\eta)(|\bar{u}|_+^2 + |\bar{v}|_+^2) + |s_0|_+^2\,|\bar{v}|_+^2\,\eta^{-1} .$$

Now $q(\eta) \to \infty$ as $\eta \to \eta_0$. We see that we can violate (4.11) by choosing ϵ small enough and η close enough to η_0.

So we have shown that if (4.8) is σ-ill-posed so is (4.1-4.3). This completes the proof of Theorem 4.1.

4.6. Variable Coefficients and Lower Order Terms.

We now extend the above results to systems with variable coefficients and lower order terms. Specifically we consider

$$(4.26)\qquad \begin{aligned} u_t &= P_0(x,y,t)u_{xx} + \sum_{i=1}^{n} P_{1i}(x,y,t)u_{xy_i} + \sum_{j,k=1}^{n} P_{2jk}(x,y,t)u_{y_jy_k} \\ &\quad + A_0(x,y,t)v_x + \sum_{k=1}^{n} A_k(x,y,t)v_{y_k} \\ &\quad + C_0(x,y,t)u_x + \sum_{k=1}^{n} C_k(x,y,t)u_{y_k} \\ &\quad + C_{11}(x,y,t)u + C_{12}(x,y,t)v + F_1(x,y,t) \\[1em] v_t &= B_0(x,y,t)u_x + \sum_{k=1}^{n} B_k(x,y,t)u_{y_k} + Q_0(x,y,t)v_x + \sum_{k=1}^{n} Q_k(x,y,t)v_{y_k} \\ &\quad + C_{21}(x,y,t)u + C_{22}(x,y,t)v + F_2(x,y,t) . \end{aligned}$$

The boundary conditions at $x = 0$ are

$$(4.27)\quad \begin{aligned} T_1(y,t)u_x + \sum_{k=1}^{n} T_{2k}(y,t)u_{y_k} + T_0(y,t)u + S_1(y,t)v &= g_1(y,t) \\ T(y,t)u \quad + \qquad\qquad\qquad\qquad S(y,t)v &= g_2(y,t)\ , \end{aligned}$$

and $u = 0,\ v = 0$ at $t = 0$.

We assume that all the coefficient matrices are C^∞ functions and tend to constant values as their arguments tend to infinity.

Define

$$\tilde{u} = e^{\eta t}\ \Sigma^{-1}(e^{-\eta t}(u_x + P_0^{-1}A_0 v)),$$

$$w = (u,\tilde{u},v)'\ .$$

Then (4.26) can be written as

$$w_x = N(x,y,t,D_y,D_t,\eta)w + N_0(x,y,t,D_y,D_t,\eta)w + F\ ,$$

where $N(x,y,t,D_y,D_t,\eta)$ is a pseudo-differential operator whose symbol is

$$\begin{pmatrix} 0 & \sigma & 0 \\ P_0^{-1}\left(\dfrac{s - P_2(\omega)}{\sigma}\right) & -iP_0^{-1}P_1\cdot\omega & 0 \\ -iQ_0^{-1}B\cdot\omega & Q_0^{-1}B_0\sigma & Q_0^{-1}(s - iQ\cdot\omega) \end{pmatrix}$$

and $N_0(x,y,t,D_y,D_t,\eta)$ is a bounded pseudo-differential operator. $s = \eta + i\tau$ where τ is the dual Fourier variable of t.

As we will show in Chapter VII, we can construct a pseudo-differential operator $R(y,t,D_y,D_t,\eta)$ whose symbol satisfies (4.18) for each value of (y,t). We will also prove the following analogue of Gårding's inequality.

Theorem 4.8. If the symbol $R(y,t,\omega,\tau,\eta)$ satisfies (4.18) for each value of (y,t), then we can modify R so that for $\eta \geq \eta_0$

$$\mathrm{Re}(w,RNw)_\eta \geq \mathrm{Re}(u,\Sigma u)_\eta + \mathrm{Re}(u_x,\Sigma^{-1}u_x)_\eta + \eta\|v\|_\eta^2 \quad . \tag{4.28}$$

And moreover, if for each (y,t) the corresponding frozen coefficient problem of (4.26) is σ-well-posed, then

$$\mathrm{Re}\ \langle w, Rw\rangle_\eta \geq -c_1(|\Sigma^{-1}g_1|_\eta^2 + |g_2|_\eta^2) + c_2|w|_\eta^2 \tag{4.29}$$

for some positive constants c_1 and c_2.

By the frozen coefficient problem of (4.26) at (y,t) we mean the system

$$u_t = P_0 u_{xx} + \Sigma P_{1i} u_{xy_i} + \Sigma P_{2jk} u_{y_j y_k}$$

$$v_t = B_0 u_x + \Sigma B_k u_{y_k} + Q_0 v_x + \Sigma Q_k v_{y_k}$$

with boundary conditions

$$T_1 u_y + \Sigma T_{2k} u_{y_k} = g_1$$

$$Tu + Sv = g_2$$

where the coefficients all are held constant at their values at $(0,y,t)$. We also need to consider the frozen, coefficient problem obtained letting $|\mathbf{y}| + t \to \infty$.

Using the above we have

$$\mathrm{Re}(u,\Sigma u)_\eta + \mathrm{Re}(u_x,\Sigma^{-1}u_x)_\eta + \eta\|v\|_\eta^2$$

$$\le \mathrm{Re}(w,\, RNw)_\eta$$

$$= \mathrm{Re}(w,Rw_x)_\eta - \mathrm{Re}(w,RN_0w)_\eta - \mathrm{Re}(w,RF)_\eta$$

$$\le -\frac{1}{2}\,\mathrm{Re}\,\langle w,Rw\rangle_\eta + C\|w\|_\eta^2 + \|\Sigma^{-1}F_1\|_\eta^2 + \|F_2\|_\eta^2$$

$$\le c_1(|\Sigma^{-1}(g_1 - T_0u - S_1v)|_\eta^2 + |g_2|_\eta^2) + C\|w\|_\eta^2 + \|\Sigma^{-1}F_1\|_\eta^2 + \|F_2\|_\eta^2$$

$$\le c_1(|\Sigma^{-1}g_1|_\eta^2 + |g_2|_\eta^2) + \|\Sigma^{-1}F_1\|_\eta^2 + \|F_2\|_\eta^2 + C(|\Sigma^{-1}u|_\eta^2 + |\Sigma^{-1}v|_\eta^2 + \|w\|_\eta^2)\ .$$

We use

$$\|w\|_\eta^2 \le C(\|u\|_\eta^2 + \|\Sigma^{-1}u_x\|_\eta^2 + \|v\|_\eta^2)$$

$$\le \frac{C}{\sqrt{\eta}}\,(\mathrm{Re}(u,\Sigma u)_\eta + \mathrm{Re}(u_x,\Sigma^{-1}u_x)_\eta) + C\|v\|_\eta^2$$

and

$$|\Sigma^{-1}u|_\eta^2 + |\Sigma^{-1}v|_\eta^2 \le \eta^{-1}(|u|_\eta^2 + |v|_\eta^2)\ .$$

Then for η_0 sufficiently large we have for all $\eta \ge \eta_0$

$$\mathrm{Re}(u,\Sigma u)_\eta + \mathrm{Re}(u_x,\Sigma^{-1}u_x)_\eta + \eta\|v\|_\eta^2 + |u|_\eta^2 + |\Sigma^{-1}u_x|_\eta^2 + |v|_\eta^2 \tag{4.30}$$

$$\le C(|\Sigma^{-1}g_1|_\eta^2 + |g_2|_\eta^2 + \|\Sigma^{-1}F_1\|_\eta^2 + \|F_2\|_\eta^2)\ .$$

We have proved:

Theorem 4.9. If for each frozen coefficient problem of (4.26) there are no eigensolutions of either parabolic or hyperbolic type, then (4.26) is σ-well-posed.

CHAPTER V

ANOTHER DEFINITION OF WELL-POSEDNESS

5.1. Strong σ-Well-posedness.

In this chapter we consider an alternate definition of well-posedness for the equations (4.1-4.3).

To motivate the discussion, we consider the heat equation on a smooth, bounded domain $\Omega \subseteq \mathbb{R}^{n+1}$

$$u_t = \Delta u + f \quad \text{on} \quad \Omega \times [0,\infty) \tag{5.1}$$
$$u = 0 \quad \text{at} \quad t = 0 .$$

There are two natural boundary conditions that are considered. The Dirichlet boundary condition

$$u = g \quad \text{on} \quad \partial\Omega \times [0,\infty) \tag{5.2}$$

and the Neumann boundary condition

$$\nabla_n u = g \quad \text{on} \quad \partial\Omega \times [0,\infty) \tag{5.3}$$

For the Neumann problem one can estimate the gradient of the solution on Ω, for example,

$$\eta\|u\|_\eta^2 + \|\nabla u\|_\eta^2 + |u|_\eta^2 \leq C(|g|_\eta^2 + \|f\|_\eta^2) . \tag{5.4}$$

But for the Dirichlet problem one cannot estimate the gradient of the solution in terms of the L^2 norms of the data.

One possible estimate for the Dirichlet problem is

$$\sqrt{\eta}\ \|u\|_\eta^2 + \|\nabla u\|_{\eta,\Omega'}^2 + |u|_\eta^2 \leq C(|g|_\eta^2 + \|f\|_\eta^2) \tag{5.5}$$

where $\Omega' \subseteq \overline{\Omega'} \subseteq \Omega$, and C depends on the distance between $\overline{\Omega'}$ and $\partial\Omega$. (Equation (5.5) follows from our results of Chapter VI, and (5.4) follows from the divergence theorem and the theorem of the trace.)

The estimates we obtained in Chapter IV arose from treating the boundary conditions as Dirichlet-type conditions. Now we examine when one can get stronger estimates involving the gradient up to the boundary. To do this we proceed as in Chapter IV, up to the point where we converted the system of ordinary differential equations in x to a first order system. Then we introduce as our new dependent variable

$$w = \begin{pmatrix} \sigma u \\ u_x \\ v \end{pmatrix} = \begin{pmatrix} u_1 \\ \tilde{u}_1 \\ v \end{pmatrix} .$$

Neglecting lower order terms we obtain the following system analogous to (4.8).

$$w_x = \begin{pmatrix} N_{11}(\omega,s) & N'_{12}(\omega,s) \\ & \\ 0 & N_{22}(\omega,s) \end{pmatrix} w + \begin{pmatrix} 0 \\ -P_0^{-1}F_1 \\ -Q_0^{-1}F_2 \end{pmatrix} \tag{5.6}$$

$$= N'(\omega,s) + F' \quad ,$$

with the boundary condition

$$\begin{pmatrix} iT_2\cdot\omega\sigma^{-1} & T_1 & S_1 \\ 0 & 0 & S \end{pmatrix} w = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

where

$$N'_{12}(\omega,s) = -\begin{pmatrix} 0 \\ P_0^{-1}A_0N_{22}(\omega,s) + iP_0^{-1}A\cdot\omega \end{pmatrix} .$$

We see that we obtain a system similar to (4.8), except that now $N'(\omega,s)$ is upper block triangular instead of lower block triangular as was $N(\omega,s)$, and similarly for the boundary operators.

We define σ-well-posedness for (5.6) as we did for (4.8), that is, (5.6) is σ-well-posed if there are constants C_0 and η_0 such that for $\eta \geq \eta_0$

$$(5.7) \qquad \begin{aligned} &\mathrm{Re}\ \sigma(\|u_1\|_+^2 + \|\tilde{u}_1\|_+^2) + \eta\|v\|_+^2 + |w|^2 \\ &\leq C_0(|g|^2 + \|F'\|_+^2) \ . \end{aligned}$$

Applying this definition to (4.1-4.3) and using our new normalization, we have

Definition 5.1. The initial boundary value problem (4.1-4.3) is strongly σ-well-posed if there are constants η_0 and C_0 such that

$$(5.8) \qquad \begin{aligned} &\mathrm{Re}(u,\Sigma|S|u)_\eta + \mathrm{Re}(\nabla u,\Sigma\nabla u)_\eta + \eta\|v\|_\eta^2 + ||S|^{1/2}u|_\eta^2 + |\nabla u|_\eta^2 \\ &\leq C_0(|g_1|_\eta^2 + |g_2|_\eta^2 + \|F_1\|_\eta^2 + \|F_2\|_\eta^2) \end{aligned}$$

for all $\eta \geq \eta_0$.

We have used the inequality

$$|s| + \omega^2 \leq \sqrt{2}\ |\sigma|^2 \leq \sqrt{2}\ (|s| + \omega^2)$$

to replace

$$\mathrm{Re}(\Sigma u, \Sigma^2 u)_\eta + \mathrm{Re}(u_x, \Sigma u_x)_\eta$$

by

$$\mathrm{Re}(u, \Sigma |s| u)_\eta + \mathrm{Re}(\nabla u, \Sigma \nabla u)_\eta \ .$$

$|S|$ is the pseudo-differential operator whose symbol is $|s|$.

We now define two more types of eigensolutions.

Definition 5.2. (u, v_0) is a strong eigensolution of parabolic type at (ω, s) if it satisfies:

(5.9) a) $(u, v_0) \neq 0$,

b) $(\omega, s) \neq 0$, $\mathrm{Re}\ s \geq 0$,

c) $su = P_0 u_{xx} + iP_1 \cdot \omega u_x + P_2(\omega) u$,

d) $v_0^+ = 0$,

e) $T_1 u_x + iT_2 \cdot \omega u + S_1 v_0 = 0$,

$Sv_0 = 0$ at $x = 0$,

f) $u \in L^2(\mathbb{R}_+)$ and $v_0 \in \mathbb{C}^q$.

Definition 5.3. (u, v) is a strong eigensolution of hyperbolic type at (ω, s) if it satisfies

(5.10) a) $(u,v) \neq 0$,

b) $\omega \neq 0$, $\mathrm{Re}\, s \geq 0$,

c) $0 = P_0 u_{xx} + iP_1 \cdot \omega u_x + P_2(\omega) u + A_0 v_x + iA \cdot \omega v$,

d) $sv = Q_0 v_x + iQ \cdot \omega v$,

e) $T_1 u_x + iT_2 \cdot \omega u + S_1 v = 0$

$Sv = 0$ at $x = 0$,

f) For $\mathrm{Re}\, s > 0$, v is in $L^2(\mathbb{R}_+)$, and for $\mathrm{Re}\, s = 0$ v is is the limit of $L^2(\mathbb{R}_+)$ solutions of d) as $\mathrm{Re}\, s \to 0$. u is in $L^2(\mathbb{R}_+)$.

Analogous to Theorem 4.5, we have

Theorem 5.1. The initial boundary value problem (4.1-4.3) is strongly σ-well-posed if and only if it has no strong eigensolutions of either parabolic or hyperbolic type.

The proof of Theorem 5.1 is analogous to that of Theorem 4.5, so we will omit it.

The next theorem expresses some relations between the two types of well-posedness considered so far and b_1, which is the number of boundary conditions involving derivatives of u, (see Assumption 4.1).

Theorem 5.2. Consider the initial boundary value problem (4.1-4.3).

a) If $b_1 > p$, it is always σ-ill-posed.

b) If $b_1 < p$, it is always strongly σ-ill-posed.

c) If $b_1 = p$, the following three statements are equivalent.

1) The problem is σ-well-posed.

2) The problem is strongly σ-well-posed.

3) The following two initial boundary value problems are well-posed.

$$(5.11) \qquad \begin{aligned} & u_t = P_0 u_{xx} + \Sigma P_{1k} u_{xy_k} + \Sigma P_{2jk} u_{y_j y_k} + F_1 \\ & T_1 u_x + \Sigma T_{2k} u_{y_k} = g_1 \qquad u = 0 \ \text{ at } \ t = 0 \\ & v_t = Q_0 v_x + \Sigma Q_k v_{y_k} + F_2 \\ & Sv = g_2 \qquad v = 0 \ \text{ at } \ t = 0 \ . \end{aligned}$$

We point out that the two systems in (5.11) are a special case of (4.1-4.3) in which all the coupling terms vanish. For the proof of Theorem 5.2, we shall regard (5.11) as a special case of (4.1-4.3) and "well-posed" for (5.11) means σ-well-posed. After the proof we shall comment on the well-posedness of each of the systems of (5.11) separately.

Proof of a) and b)

If $p < b_1$ then $q^- > b_2$, hence we can find an eigensolution of parabolic type at any (ω, s), by choosing $u = 0$. For then, we need only solve

$$Sv_0 = 0, \qquad v_0^+ = 0 \ .$$

This means we have less than q^- conditions on the q^- non-zero components of v_0, and there is always a non-trivial solution.

If $p > b_1$, there is always a strong eigensolution of hyperbolic type. If we set $v = 0$, we must solve

$$P_0 u_{xx} + iP_1 \cdot \omega u_x + P_2(\omega) u = 0$$

$$T_1 u_x + iT_2 \cdot \omega u = 0 \qquad \text{for } u \in L^2(\mathbb{R}_+).$$

But there are less than p boundary conditions and the solution space of the differential equation is of dimension p; so there is always a non-trivial solution.

Part c) of Theorem 5.2 is implied by the following two propositions.

Proposition 5.1. If $b_1 = p$, the following three statements are equivalent.

1) There are no strong eigensolutions of parabolic type for (4.1-4.3).

2) There are no eigensolutions of parabolic type for (4.1-4.3).

3) There are no eigensolutions of parabolic type for (5.11).

Proposition 5.2. If $b_1 = p$, the following three statements are equivalent.

1) There are no strong eigensolutions of hyperbolic type for (4.1-4.3).

2) There are no eigensolutions of hyperbolic type for (4.1-4.3).

3) There are no eigensolutions of hyperbolic type for (5.11).

Proof of Proposition 5.1.

The boundary operators, in each case, map the $p+q^-$ dimensional space of solutions of

$$P_0 u_{xx} + iP_1 \cdot \omega u_x + (P_2(\omega) - s)u = 0, \qquad u \in L^2(\mathbb{R}_+)$$

$$v_0^+ = 0, \qquad v_0 \in \mathbb{C}^q$$

into $\mathbb{C}^{p+q^-}$. The condition $b_1 = p$ is equivalent to each of the following statements.

1) For strong eigensolutions of parabolic type for (4.1-4.3), the boundary operator is block upper triangular.

2) For eigensolutions of parabolic type for (4.1-4.3), the boundary operator is block lower triangular.

3) For eigensolutions of parabolic type for (5.11), the boundary operator is block diagonal.

In each case the two blocks on the main diagonal are represented by $T_1u_x + iT_2\cdot\omega u$ and Sv_0. The nonexistence of eigensolutions says that the boundary operator is nonsingular. This is then equivalent to each of the main diagonal blocks being nonsingular. This proves Proposition 5.1.

The proof of Proposition 5.2 is similar to the above and will not be given.

5.2. Parabolic and Hyperbolic Systems

The following theorems for the initial boundary value problem for parabolic and hyperbolic systems are contained in the preceding work. We state them here for the sake of completeness.

Theorem 5.3. The parabolic initial boundary value problem on $\mathbb{R}^{n+1}_+$

$$u_t = P_0 u_{xx} + \sum_{k=1}^{n} P_{1k} u_{xy_k} + \sum_{j,k=1}^{n} P_{2jk} u_{y_j y_k} + F(x,y,t)$$

$$T_1 u_x + \sum_{k=1}^{n} T_{2k} u_{y_k} = g_1(y,t)$$

$$Tu = g_2(y,t) \qquad \text{at} \quad x = 0$$

$$u = 0 \qquad \text{at} \quad t = 0$$

is well-posed if and only if there are no nontrivial $L^2(\mathbb{R}_+)$ solutions to the following ordinary differential equation in $\mathbb{R}_+$.

For $(\omega,s) \neq 0$, and $\text{Re } s \geq 0$:

$$s\hat{u} = P_0 \hat{u}_{xx} + iP_1 \cdot \omega \hat{u}_x + P_2(\omega)\hat{u}$$

$$T_1 \hat{u}_x = iT_2 \cdot \omega \hat{u} = 0$$

$$T\hat{u} = 0$$

If it is well-posed, we have for $\eta \geq \eta_0$

$$\text{Re}(u,\Sigma u)_\eta + \text{Re}(u_x,\Sigma^{-1}u_x)_\eta + |u|_\eta^2 + |\Sigma^{-1}u_x|_\eta^2$$
$$\leq C_0(|\Sigma^{-1}g_1|_\eta^2 + |g_2|_\eta^2 + \|\Sigma^{-1}F\|_\eta^2) .$$

Also if $T = 0$, we have

$$\text{Re}(u,\Sigma|S|u)_\eta + \text{Re}(\nabla u,\Sigma\nabla u)_\eta + ||S|^{1/2}u|_\eta^2 + |\nabla u|_\eta^2 \leq C_0(|g_1|_\eta^2 + \|F\|_\eta^2).$$

We assume in the above that there are precisely p boundary conditions, where p is the dimension of u.

The above condition for well-posedness is essentially that given by Ladyženskaja [8].

Theorem 5.4 (Kreiss [6], Ralston [15]). The strictly hyperbolic initial boundary value problem on $\mathbb{R}_+^{n+1}$

$$v_t = Q_0 v_x + \sum_{k=1}^{n} Q_k v_{y_k} + F(x,y,t)$$

$$Sv = g \qquad \text{at} \quad x = 0$$

$$v = 0 \qquad \text{at} \quad t = 0$$

is well-posed if and only if the ordinary differential equation on $\mathbb{R}_+$

$$s\hat{v} = Q_0 \hat{v}_x + iQ\cdot\omega\hat{v}$$

$$S\hat{v} = 0$$

has no nontrivial $L^2(\mathbb{R}_+)$ solutions for $\operatorname{Re} s > 0$, and for $\operatorname{Re} s = 0$, there are no $L^2(\mathbb{R}_+)$ solutions v^ϵ to

$$(s + \epsilon)v = Q_0 v_x + iQ\cdot\omega v$$

such that

$$Sv^\epsilon \to 0 \quad \text{and} \quad v^\epsilon \not\to 0 \quad \text{as} \quad \epsilon \to 0.$$

If it well-posed, we have the estimate

$$\eta\|v\|_\eta^2 + |v|_\eta^2 \le C_0(|g|_\eta^2 + \|F\|_\eta^2) , \qquad \eta \ge \eta_0 .$$

It is possible to generalize the results of Section 1 by decomposing u as (u^1,u^2) in which the estimates on u^1 are of the strong type and those on u^2 are of the nonstrong type. However this requires that the operator $P(D)$ decomposes in an appropriate fashion. Since we have not restricted $P(D)$ in any way beyond the parabolicity assumption, the generalization does not seem appropriate here.

CHAPTER VI

THE INITIAL BOUNDARY VALUE PROBLEM FOR SMOOTH BOUNDED DOMAINS

We now consider the initial boundary value problem for incompletely parabolic systems. We rewrite (1.1) as

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} P(z,t,D) & A(z,t,D) \\ B(z,t,D) & Q(z,t,D) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + F(z,t) \tag{6.1}$$

for $z \in \Omega \subseteq \mathbb{R}^{n+1}$, $t \geq 0$.

We have the boundary conditions

$$T(z',t,D)u + S(z',t)v = g(z',t) \tag{6.2}$$

for $z' \in \partial\Omega$, $t \geq 0$, and initial data

$$u = 0, \qquad v = 0 \qquad \text{at} \quad t = 0\,. \tag{6.3}$$

The system

$$u_t = P(z,t,D)u$$

is a second order Petrovskii parabolic system, and the system

$$v_t = Q(z,t,D)v$$

is a first order strictly hyperbolic system. $A(z,t,D)$, $B(z,t,D)$ and $T(z',t,D)$ are first order differential operators. All the coefficients are assumed to be C^∞ and tend to constants as $t \to \infty$. We also assume that Ω is a bounded open set with C^∞ boundary.

<u>Definition 6.1</u>. The frozen coefficient initial boundary value problem for (6.1-6.3) at (z_0', t_0), $z_0' \in \partial\Omega$, $t_0 \geq 0$, is the initial boundary value problem

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} P(z_0',t_0,D) & A(z_0',t_0,D) \\ B(z_0',t_0,D) & Q(z_0',t_0,D) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + F(x,y,t)$$

with boundary conditions

$$T(z_0',t_0,D)u + S(z_0',t_0)v = g(x,y,t)$$

and initial conditions $u = 0$, $v = 0$ at $t = 0$. This is defined on the half-space $\mathbb{R}_+^{n+1}(z_0')$ where the ray $x \geq 0$, $y = 0$ is the inward normal ray to $\partial\Omega$ at z_0' and the space $x = 0$ is the tangent n-space to $\partial\Omega$ at z_0'.

The limiting values of the coefficients as $t \to \infty$ must also be considered, so we will allow $t = \infty$ as well. We set $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$.

We now show that by examining the frozen coefficient problems via Theorem 4.5 we can decide if the initial boundary value problem (6.1-6.3) is well-posed.

<u>Theorem 6.1</u>. If the frozen coefficient initial boundary value problem at (z',t) is σ-well-posed for each $(z',t) \in \partial\Omega \times \bar{\mathbb{R}}_+$, then the initial boundary value problem (6.1-6.3) is well-posed.

Proof.

We take a finite open covering $\{U^\alpha\}$ of Ω, and a partition of unity $\{\varphi^\alpha\}$ such that

(6.4) a) supt $\varphi^\alpha \subseteq U^\alpha$

b) $\sum_\alpha |\varphi^\alpha(z)|^2 = 1$

c) If $U^\alpha \cap \partial\Omega \neq \emptyset$ then there is a C^∞ map $\psi^\alpha : U^\alpha \to \mathbb{R}^{n+1}_+$ such that $\psi^\alpha(U^\alpha \cap \partial\Omega) \subseteq \mathbb{R}^n$.

Set $f^\alpha(z,t) = \varphi^\alpha(z)\, f(z,t)$ for any function $f(z,t)$. Then we have

$$(6.5)\quad \begin{pmatrix} u^\alpha \\ v^\alpha \end{pmatrix}_t = \begin{pmatrix} P(z,t,D) & A(z,t,D) \\ B(z,t,D) & Q(z,t,D) \end{pmatrix} \begin{pmatrix} u^\alpha \\ v^\alpha \end{pmatrix}$$

$$+ \begin{pmatrix} \tilde{P}_\alpha(z,t,D) & \tilde{A}_\alpha(z,t) \\ \tilde{B}_\alpha(z,t) & \tilde{Q}_\alpha(z,t) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + F^\alpha(z,t)$$

where $\tilde{P}_\alpha$ is a first order differential operator. When $U^\alpha \cap \partial\Omega \neq \emptyset$ we have the boundary conditions

$$T(z',t,D)u^\alpha + S(z',t)v^\alpha = g^\alpha(z,t) - \tilde{T}_\alpha(z',t)u\ .$$

There are now two cases to consider, depending on whether $U^\alpha \cap \partial\Omega = \emptyset$ or not. In the case where $U^\alpha \cap \partial\Omega = \emptyset$, we treat (6.5) as a Cauchy problem for (u^α, v^α). Notice that we can alter the coefficients on the exterior of U^{α} so that they tend to constants as $t + |z| \to \infty$. Then from (3.11) we have

$$(6.6) \qquad \eta(\|u^{\alpha}\|_{\eta}^{2} + \|v^{\alpha}\|_{\eta}^{2}) + \|\nabla u^{\alpha}\|_{\eta}^{2}$$

$$\leq \|F^{\alpha}\|_{\eta}^{2} + C(\|u^{\alpha}\|_{\eta}^{2} + \|v^{\alpha}\|_{\eta}^{2}) + C(\|u\|_{\eta,U^{\alpha}}^{2} + \|v\|_{\eta,U^{\alpha}}^{2})$$

$$- \operatorname{Re}(u^{\alpha}, H\tilde{P}_{\alpha} u)_{\eta}$$

The last term is estimated as follows. If $R(z,t,D)$ is a bounded pseudo-differential operator, then

$$(u^{\alpha}, RD_{z_i} u)_{\eta} = (\varphi^{\alpha} u, RD_{z_i} u)_{\eta}$$

$$= (u, RD_{z_i} u^{\alpha})_{\eta} + (u, R'u)_{\eta}$$

where R' is a bounded pseudo-differential operator. So

$$(u^{\alpha}, RD_{z} u) \leq C\|u\|_{\eta,U^{\alpha}}^{2} + \epsilon\|\nabla u^{\alpha}\|_{\eta}^{2}$$

Then using (6.6) we have, for η large enough,

$$(6.7) \qquad \eta\|w^{\alpha}\|_{\eta}^{2} + \|\nabla u^{\alpha}\|_{\eta}^{2} \leq C(\|F^{\alpha}\|_{\eta}^{2} + \|w\|_{\eta,U^{\alpha}}^{2})$$

Now consider the case where $U^{\alpha} \cap \partial\Omega \neq \emptyset$. We change coordinates on U^{α} via the map ψ^{α}. Let

$$\bar{u} = u^{\alpha} \circ \psi^{\alpha}\,, \qquad \bar{v} = v^{\alpha} \circ \psi^{\alpha}\,, \qquad \text{etc.}$$

and

$$V^{\alpha} = \psi^{\alpha}(U^{\alpha} \cap \Omega) \qquad \bar{V}^{\alpha} = \psi^{\alpha}(U^{\alpha} \cap \partial\Omega)$$

and using an abuse of notation, let

$$u = u_{|U^{\alpha}} \circ \psi^{\alpha}\,, \qquad v = v_{|U^{\alpha}} \circ \psi^{\alpha}\,.$$

Then, letting (x,y) be the new coordinates, $x \geq 0$, $y \in \mathbb{R}^n$, we have the initial boundary value problem

$$\text{(6.8)} \qquad \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}_t = \begin{pmatrix} \bar{P}(x,y,t,D) & \bar{A}(x,y,t,D) \\ \bar{B}(x,y,t,D) & \bar{Q}(x,y,t,D) \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$$

$$+ \begin{pmatrix} \tilde{P}(x,y,t,D) & \tilde{A}(x,y,t) \\ \tilde{B}(x,y,t) & \tilde{Q}(x,y,t) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \bar{F}(x,y,t)$$

and the boundary conditions

$$T_1(y,t)\bar{u}_x + \sum T_{2k}(y,t)\bar{u}_y + S_1(y,t)\bar{v} = \bar{g}_1(y,t) + \tilde{T}(y,t)u$$
$$T(y,t)\bar{u} + S(y,t)\bar{v} = \bar{g}_2(y,t)\,.$$

We can extend $\bar{u}, \bar{v}, \bar{F}$, and $\bar{y}$ by defining them to be zero outside V^{α}. Similarly for the coefficients in the second operator on the right. The coefficients of the first operator on the right and the boundary condition coefficients can be extended to all of $\mathbb{R}^{n+1}_{+}$ and $\mathbb{R}^n$, respectively, so that they tend to constants at infinity, all our assumptions are satisfied, and the initial boundary value problem remains σ-well-posed.

Since (6.8) is σ-well-posed we have the following estimate on $\mathbb{R}_+^{n+1}$ from (4.11)

$$(6.9) \qquad \mathrm{Re}(u, \Sigma\bar{u})_\eta + \mathrm{Re}(\bar{u}_x, \Sigma^{-1}\bar{u}_x)_\eta + \eta\|\bar{v}\|_\eta^2 + |\bar{w}|_\eta^2$$

$$\leq C[|\Sigma^{-1}\bar{g}_1|_\eta^2 + |\bar{g}_2|_\eta^2 + |\Sigma^{-1}w|^2_{\eta,\bar{V}^\alpha} + \|\Sigma^{-1}\bar{F}_1\|_\eta^2 + \|w\|^2_{\eta,V^\alpha}$$

$$+ \|\bar{F}_\alpha\|_\eta^2 - \mathrm{Re}(\bar{w}, R\Sigma^{-1}\tilde{P}u)_\eta].$$

To estimate the last term we use the following inequality. If $G(x,y,t,D_y,D_t,\eta)$ is a bounded pseudo-differential operator (with parameter η) with support in V^α, then

$$(\bar{w}, G\Sigma^{-1}u_{y_i})_\eta \leq C_0(\|\bar{w}\|_\eta^2 + \|u\|^2_{\eta,V^\alpha})$$

and

$$(\bar{w}, G\Sigma^{-1}u_x)_\eta = (\varphi^\alpha \circ \psi^\alpha w, G\Sigma^{-1}u_x)_\eta$$

$$= (w, G\Sigma^{-1}\bar{u}_x)_\eta + (w, G'\Sigma^{-1}(\varphi^\alpha_x \circ \psi^\alpha)_x u)_\eta$$

$$\leq C(\|w\|^2_{\eta,V^\alpha} + \|\Sigma^{-1}\bar{u}_x\|_\eta^2)$$

$$\leq C(\|w\|^2_{\eta,V^\alpha} + \eta^{-1/2}\,\mathrm{Re}(\bar{u}_x, \Sigma^{-1}\,\bar{u}_x)_\eta)\,.$$

Also, $|\Sigma^{-1}w|^2_{\eta,V^\alpha} \leq \eta^{-1}|w|^2_{\eta,V^\alpha}$. So, for η sufficiently large, we have from the above estimates that

(6.10) $$\sqrt{\eta}\,\|\bar{u}\|_\eta^2 + \eta\|\bar{v}\|_\eta^2 + |\bar{w}|_\eta^2 \le C(|\bar{g}|_\eta^2 + \|F\|_\eta^2 + \|w\|^2_{\eta,V^\alpha} + \frac{1}{\eta}\,|w|^2_{\eta,V^\alpha}) .$$

On $\Omega \cap \operatorname{supt} \varphi^\alpha$ we have

(6.11) $$\sqrt{\eta}\,\|\varphi^\alpha u\|^2_{\eta,\Omega} + \eta\|\varphi^\alpha v\|^2_{\eta,\Omega} + |\varphi^\alpha w|^2_{\eta,\partial\Omega}$$

$$\le C(|\varphi^\alpha g|^2_{\eta,\partial\Omega} + \|\varphi^\alpha F\|^2_{\eta,\partial\Omega} + \|w\|^2_{\eta,\Omega} + \frac{1}{\eta}\,|w|^2_{\eta,\partial\Omega}) .$$

Summing (6.11) and (6.7) over all α and using (6.4) we obtain, for η sufficiently large,

(6.12) $$\sqrt{\eta}\,\|w\|_{\eta,\Omega} + |w|^2_{\eta,\partial\Omega} + \|\nabla u\|^2_{\eta,\Omega'} \le C(|g|^2_{\eta,\partial\Omega} + \|F\|^2_{\eta,\Omega})$$

where $\Omega' \subseteq \overline{\Omega'} \subseteq \Omega$ and Ω' is open.

This proves Theorem 6.1.

Similarly we can prove,

<u>Theorem 6.2</u>. If the frozen coefficient initial boundary value problem for (z',t) is strongly σ-well-posed at each boundary point $(z',t) \in \partial\Omega \times \overline{R}_+$, then we have the following estimate

(6.13) $$\eta\|w\|^2_{\eta,\Omega} + \|\nabla u\|^2_{\eta,\Omega} + |w|^2_{\eta,\partial\Omega} \le C(|g|^2_{\eta,\partial\Omega} + \|F\|^2_{\eta,\Omega}) .$$

The necessary and sufficient conditions for the frozen coefficient problem to be strongly σ-well-posed have been given in Theorem 5.1.

We should point out that the system

$$v_t = Q(z,t,D)v \tag{6.14}$$

can satisfy a weaker assumption than strict hyperbolicity and still have the results of this chapter hold. The estimates for the solution of (6.1-6.3) in the interior of Ω require only that (6.14) be hyperbolic. On the boundary of Ω, the estimates depend on the existence of the matrix $R(\omega,s)$ of Theorem 4.7. Agranovich [2], Taniguchi [18], and Majda and Osher [10, page 618] have all given conditions implying the existence of $R(\omega,s)$ which are weaker than strict hyperbolicity. If (6.14) is hyperbolic on $\Omega \times \mathbb{R}_+$ and satisfies one of the above conditions at each point of $\partial\Omega \times \overline{\mathbb{R}}_+$ then Theorems 6.1 and 6.2 remain valid.

CHAPTER VII

PSEUDO-DIFFERENTIAL OPERATORS WITH PARAMETER

In this chapter we present a theory of pseudo-differential operators that depend upon a parameter. Our theory will be analogous to the usual theory of pseudo-differential operators and many results for the operators with parameter will follow immediately from the usual theory. Our presentation will follow Taylor [19], also see Nirenberg [12].

We shall then prove Theorem 4.7 by constructing the pseudo-differential operator $R(\omega,\tau,\eta)$ which depends on the parameter η.

Finally, we use our theory to prove analogues of Gårding's inequality. These inequalities are used to prove Theorem 4.8.

7.1. Definitions and Formulae

Let N be an interval in $\mathbb{R}_+$, N will be the set of parameters. For $\xi \in \mathbb{R}^n$ and $\eta \in N$, define

$$\langle \xi,\eta\rangle = (\xi^2 + \eta^2)^{1/2} .$$

Definition 7.1. For $m, \rho \in \mathbb{R}$, $0 < \rho \leq 1$, Sp_ρ^m is the set of $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times N)$ with the property that for any multi-indices α and β there is a constant $C_{\alpha,\beta}$ such that

$$|D_z^\alpha D_\xi^\beta p(z,\xi,\eta)| \leq C_{\alpha,\beta} \langle \xi,\eta\rangle^{m-\rho|\beta|} .$$

If $p \in Sp_\rho^m$ we say p is a symbol with parameter of order m and type ρ.

We note that the usual theory of pseudo-differential operators corresponds to choosing $N = \{\eta_0\}$, $\eta_0 \neq 0$. For us N will be of the form $[\eta_0,\infty)$, $\eta_0 > 0$. We shall allow $p(z,\xi,\eta)$ to be a matrix, and in this case we shall say $p \in Sp_\rho^m$ if each element of p is in Sp_ρ^m.

The Fourier transform of $u(z)$ will be

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot z} u(z)\, dz .$$

For two vector-valued functions $u(z)$, $v(z)$ we define

$$(u,v) = \int_{\mathbb{R}^n} u(z)^t v(z)\, dz$$

$$= \int_{\mathbb{R}^n} \hat{u}(\xi)^t \hat{v}(\xi)\, d\xi .$$

In this chapter we define the norms $|u|_r$ by

$$|u|_r^2 = \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \langle\xi,\eta\rangle^{2r}\, d\xi .$$

Note that for $r < 0$ and $\eta > 0$

$$|u|_r \leq \eta^r |u|_0 .$$

<u>Definition 7.2</u>. If $p(z,\xi,\eta) \in Sp_\rho^m$ then we define the pseudo-differential operator $p(z,D,\eta)$ by

$$p(z,D,\eta)\, u(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} p(z,\xi,\eta)\, e^{iz\cdot\xi}\, \hat{u}(\xi)\, d\xi \ .$$

We then write $p(z,D,\eta) \in PSp(m,\rho)$ and $p(z,\xi,\eta)$ is the symbol of $p(z,D,\eta)$.

Notice that a differential operator of order m is in $PSp(m,1)$.

If $p(z,D,\eta) \in PSp(m,\rho)$ then, for each fixed non-zero value of $\eta \in N$, $p(z,D,\eta)$ can be considered as a pseudo-differential operator according to the usual theory. Therefore, all of the elementary properties of pseudo-differential operators carry over to pseudo-differential operators with parameter. In particular, the asymptotic expansion formulae for adjoints and products are exactly the same.

For convenience we collect here the formulae we will need.

If $p(z,\xi,\eta) \in Sp_\rho^m$, $\{p_\alpha(z,\xi,\eta)\}_{\alpha \geq 0}$ is a set of symbols such that $p_\alpha(z,\xi,\eta) \in Sp\rho^{m-|\alpha|\rho}$ (α is a multi-index), and

$$p(z,\xi,\eta) - \sum_{0 \leq |\alpha| < N} p_\alpha(z,\xi,\eta) \in Sp_\rho^{m-N\rho}$$

for every $N > 0$, then we write

$$p(z,\xi,\eta) \sim \sum_{\alpha \geq 0} p_\alpha(z,\xi,\eta)$$

and call it an asymptotic expansion for $p(z,\xi,\eta)$.

$P^*(z,D,\eta)$ is the adjoint of $P(z,D,\eta)$ defined by

$$\int (P^* u)^t v \, dz = \int u^t (Pv) dz \ .$$

Its symbol is given by

$$P^*(z,\xi,\eta) \sim \sum_{\alpha \geq 0} i^{|\alpha|} \frac{1}{\alpha!} D_\xi^\alpha D_z^\alpha P(z,\xi,\eta)^t \ . \tag{7.1}$$

If $P(z,D,\eta) \in PSp(m_1,\rho)$ and $Q(z,D,\eta) \in PSp(m_2,\rho)$, then the composition $P \cdot Q(z,D,\eta)$ is in $PSp(m_1 + m_2,\rho)$ and its symbol is given by

$$P \cdot Q(z,\xi,\eta) \sim \sum_{\alpha \geq 0} i^{|\alpha|} \frac{1}{\alpha!} D_\xi^\alpha P(z,\xi,\eta) \ D_z^\alpha Q(z,\xi,\eta).$$

In our applications of this theory we will take $z = (y,t)$, $\xi = (\omega,\tau)$, and the functions we deal with will be zero for $t \leq 0$.

For $P(y,t,D_y,D_t,\eta)$, a pseudo-differential operator with parameter, we have

$$\|e^{\eta t} P(y,t,D_y,D_t,\eta)(e^{-\eta t} u)\|_\eta^2$$

$$= \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |P(y,t,D_y,D_t,\eta)(e^{-\eta t} u(y,t))|^2 dy \, dt$$

and

$$P(y,t,D_y,D_t,\eta)(e^{-\eta t} u(y,t))$$

$$= \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} P(y,t,\omega,\tau,\eta) \ \hat{u}(\omega, \ \eta + i\tau) \ d\omega \ d\tau \ .$$

$\hat{u}$ denotes the Fourier transform of u in y and the Laplace transform in t, the dual variables being ω and $\eta + i\tau$, respectively.

In the proof of Gårding's inequality we will need to use double symbols.

Definition 7.3. $p(\xi_2, z, \xi_1, \eta) \in Sp_\rho^{m_1, m_2}$ if for all multi-indices α, β_1, and β_2 there is a constant $c_{\alpha,\beta_1,\beta_2}$ such that

$$|D_{\xi_2}^{\beta_2} D_z^{\alpha} D_{\xi_1}^{\beta_1} p(\xi_2, z_1, \xi_1, \eta)| \leq c_{\alpha,\beta_1,\beta_2} \langle \xi_2, \eta \rangle^{m_2 - \rho|\beta_2|} \langle \xi_1, \eta \rangle^{m_1 - \rho|\beta_1|} .$$

Definition 7.4. The operator $p(D,z,D,\eta)$ is defined by

$$\widehat{(P(D,z,D,\eta)u)}(\xi_2)$$

$$= (2\pi)^{-n/2} \iint p(\xi_2, z, \xi_1, \eta) e^{iz\cdot(\xi_2 - \xi_1)} \hat{u}(\xi_1)\, d\xi_1\, dz .$$

$P(D,z,D,\eta)$ can be thought of as first performing the ξ_1 differentiation, then the z multiplication, and finally the ξ_2 differentiation.

It can be shown that $P(D,z,D,\eta)$ is a pseudo-differential operator in $PSp(m_1+m_2,\rho)$ with symbol $\bar{p}(z,\xi,\eta)$ where

$$(7.2) \qquad \bar{p}(z,\xi,\eta) \sim \sum_{\alpha \geq 0} i^{|\alpha|} \frac{1}{\alpha!} D_z^{\alpha} D_{\xi_2}^{\alpha} p(\xi_2, z, \xi, \eta)\Big|_{\xi_2 = \xi} .$$

We now give a few examples that will be used later. Let $\xi = (\omega, \tau)$ then

$$\sigma = (\omega^2 + i\tau + \eta)^{1/2} \in Sp^1_{1/2},$$

$$\sigma^{-1} \in Sp^{-1/2}_{1/2} \ ,$$

$$\eta \in Sp^1_1 \ .$$

We will also need the following lemma.

Lemma 7.1. If $p(z,\omega,\tau,\eta)$ satisfies $p(z,\rho\omega,\rho^2\tau, \rho^2\eta) = \rho^m p(z,\omega,\tau,\eta)$ for $\rho \geq 1$ and $|\omega|^2 + |\tau| + \eta$ large, then $p(z,\omega,\tau,\eta) \in Sp^m_{1/2}$ if $m \geq 0$ and $p(z,\omega,\tau,\eta) \in Sp^{m/2}_{1/2}$ if $m < 0$.

Proof. By induction, we see that

$$(D^\alpha_\omega D^\beta_\tau p)(z,\rho\omega,\rho^2\tau,\rho^2\eta) = \rho^{m-|\alpha|-2|\beta|} \, D^\alpha_\omega D^\beta_\tau p(z,\omega,\tau,\eta) \ ,$$

also, for $\omega^2 + |\tau| + \eta$ large

$$(\omega^2 + |\tau| + \eta)^{1/2} \leq \langle \xi,\eta \rangle \leq (\omega^2 + |\tau| + \eta) \ .$$

We then have

$$|D^\alpha_\omega D^\beta_t \ p(z,\omega,\tau,\eta)| \leq C_{\alpha,\beta} (\omega^2 + |\tau| + \eta)^{(m-|\alpha|-2|\beta|)/2} .$$

The lemma follows easily.

7.2. The Construction of $R(y,t,\omega,s)$

We now prove Theorem 4.7 by constructing the pseudo-differential operator $R(y,t,\omega,s)$. We let $z = (y,\tau)$, $\xi = (\omega,\tau)$ and $\sigma_0 = \text{Re } \sigma$. The theorem we shall prove is actually stronger than Theorem 4.7.

Theorem 7.1. There exists a hermitian symbol $R(z,\xi,\eta) \in Sp^0_{1/2}$ satisfying the following:

(7.3) a) $H(z,\xi,\eta) = \text{Re } R(z,\xi,\eta)\, N(z,\xi,\eta) \geq \Delta(\xi,\eta)$ where

$$\Delta(\xi,\eta) = \begin{pmatrix} \sigma_0 I_{2p} & 0 \\ 0 & \eta I_q \end{pmatrix} .$$

b) $w^t \ Rw \geq c_0(\delta |w^+(\xi,\eta)|^2 - |w^-(\xi,\eta)|^2)$
where δ is a positive constant which may be taken arbitrarily large and $c_0 > 0$.

c) If $R = \begin{pmatrix} R_{11} & R^t_{21} \\ R_{21} & R_{22} \end{pmatrix}$ where R_{11} is $2p \times 2p$ and R_{22} is $q \times q$ then R_{11} and $R_{21} \in Sp^0_{1/2}$ and $R_{22} \in Sp^0_1$.

d) If $H = \begin{pmatrix} H_{11} & H^t_{21} \\ H_{21} & H_{22} \end{pmatrix}$ is a decomposition as in (c) then H_{11} and $H_{21} \in Sp^1_{1/2}$, $H_{22} \in Sp^1_1$, and $H_{21}(z,\xi,\eta) = 0$ for $\tau^2 + \eta^2 \geq c_1\omega^2$ and $\tau^2 + \eta^2 \geq c_2$ for some constants c_1, c_2.

We will need the following lemma in the proof of Theorem 7.1.

Lemma 7.2. Given the lower triangular $k \times k$ matrix $A = (a_{ij})$ such that $\operatorname{Re} a_{ii} \neq 0$, there exists a real diagonal matrix D such that

$$\operatorname{Re} DA \geq \frac{1}{2} DA_d > 0$$

where A_d is the diagonal matrix $\operatorname{diag}(\operatorname{Re} a_{11}, \ldots, \operatorname{Re} a_{kk})$.

Proof. For $u \in \mathbb{C}^k$ and any diagonal matrix D such that $DA_d > 0$ we have

$$\operatorname{Re} u^t DAu = \sum_{i=1}^{k} d_i \operatorname{Re} a_{ii} |u_i|^2 + \sum_{i=1}^{k} \sum_{j=1}^{i-1} d_i \operatorname{Re}(\bar{u}_i a_{ij} u_j)$$

$$\geq \sum_{i=1}^{k} d_i \operatorname{Re} a_{ii} |u_i|^2$$

$$- \sum_{i=1}^{k} \sum_{j=1}^{i-1} \left(\frac{(i-1) d_i |a_{ij}|^2}{\operatorname{Re} a_{ii}} |u_j|^2 + \frac{d_i \operatorname{Re} a_{ii}}{4(i-1)} |u_i|^2 \right)$$

$$= \frac{3}{4} \sum_{i=1}^{k} d_i \operatorname{Re} a_{ii} |u_i|^2 - \sum_{i=1}^{k} |u_i|^2 \sum_{j=i+1}^{k} \frac{(j-1) d_j |a_{ji}|^2}{\operatorname{Re} a_{jj}} .$$

So if we choose $d_k = \operatorname{sgn}(\operatorname{Re} a_{kk})$, and then having chosen d_j for $j > i$ set

$$d_i = \frac{1}{4} \frac{1}{\operatorname{Re} a_{ii}} \sum_{j=i+1}^{k} \frac{(j-1) d_j}{\operatorname{Re} a_{jj}} |a_{ji}|^2 ,$$

then $D = \operatorname{diag}(d_1, \ldots, d_k)$ satisfies the condition stated in the lemma.

Proof of Theorem 7.1. Fix z at some value z_0; we now construct $R(z_0,\xi,\eta) = R(\xi,\eta)$. Set $\beta = \langle\xi,\eta\rangle^{-1}$, $\xi' = \beta\xi$, $\eta' = \beta\eta$, and $N'(\xi',\eta',\beta) = \beta N(\xi,\eta)$.

$R(\xi',\eta',\beta)$ will be a hermitian matrix satisfying

$$\text{(7.4)} \quad \text{a)} \quad \operatorname{Re} RN' \geq C_0 \begin{pmatrix} \sigma_0' I_{2p} & 0 \\ 0 & \eta' I_q \end{pmatrix},$$

where $\sigma_0' = \beta\sigma_0 = \operatorname{Re}(\omega'^2 + i\beta\tau' + \beta\eta')^{1/2}$,

b) $w^t Rw \geq c(\delta|w^+(\xi',\eta',\beta)|^2 - |w^-(\xi',\eta',\beta)|^2)$ where $w^{\pm}(\xi',\eta',\beta) \in E^{\pm}(N'(\xi',\eta',\beta))$ (see Definition 4.7) and δ may be taken arbitrarily large.

We will construct $R(\xi',\eta',\beta)$ in a neighborhood of each point (ξ_0',η_0',β_0) where

$$|\xi_0'|^2 + |\eta_0'|^2 = 1 , \qquad 0 \leq \beta_0 \leq \eta_0'/\eta_0 \leq 1 .$$

For $\beta_0 > 0$ we construct $R(\xi',\eta',\beta)$ as follows. Since the eigenvalues of $N(\xi',\eta',\beta)$ are bounded away from the imaginary axis (Theorems 4.3 and 4.4), there is a transformation $U(\xi',\eta',\beta)$ which is analytic in a neighborhood of (ξ_0',η_0',β_0) such that

$$U(\xi',\eta',\beta)N'(\xi',\eta',\beta)\, U^{-1}(\xi',\eta',\beta) = \tilde{N}'(\xi',\eta',\beta) = \begin{pmatrix} N_-' & 0 \\ 0 & N_+' \end{pmatrix} .$$

N' is lower triangular and the eigenvalues of N'_- (resp. N'_+) have negative (resp. positive) real parts.

Now we choose D_+ and D_- according to Lemma 7.2 so that

$$\operatorname{Re} D_{\pm} N'_{\pm} \geq D_{\pm} (N'_{\pm})_d \, .$$

Set

$$R(\xi',\eta',\beta) = U^t(\xi',\eta',\beta) \begin{pmatrix} D_- & 0 \\ 0 & \delta D_+ \end{pmatrix} U(\xi',\eta',\beta).$$

Noting that

$$Uw = \begin{pmatrix} w^- \\ w^+ \end{pmatrix}$$

we see that (7.4a) and (7.4b) are satisfied.

In the case where $\beta_0 = 0$, $\eta'_0 \neq 0$, $\omega'_0 \neq 0$, we again have the eigenvalues of $N'(\xi'_0,\eta'_0,0)$ bounded away from the imaginary axis (Theorem 4.3). Moreover $N'(\xi',\eta',\beta)$ can be expressed as an analytic function of (ξ',η',β) in a neighborhood of $(\xi'_0,\eta'_0,0)$. We then construct $R(\xi',\eta',\beta)$ as above.

We now construct $R(\xi',\eta',\beta)$ in a neighborhood of $\xi' = (0,\tau')$, $\beta = 0$. Note that $N'_{22}(\xi',\eta',\beta) = N_{22}(\xi',\eta')$. Let $R_{22}(\xi',\eta')$ be the matrix as constructed by Kreiss [6] (see also Ralston [15]), satisfying

(7.5) a) $\operatorname{Re} R_{22} N_{22} \geq \eta'$,

b) $v^t R_{22} v \geq c_0(\delta|v^+|^2 - |v^-|^2)$.

At $\omega' = 0$, $\beta = 0$ both N'_{11} and N'_{21} are zero, but $N'_{22} = (\eta' + i\tau')Q_0^{-1}$ is non-singular. So for ω' and β sufficiently small the matrix $Y(\xi',\eta',\beta)$ defined by

$$(7.6) \qquad Y N'_{11} - N'^{t}_{22} Y = - R_{22} N'_{21}$$

is well-defined by Lemma 2.1. Moreover, $Y(\xi',\eta',\beta)$ satisfies

$$(7.7) \qquad \|Y\| \leq C\|N'_{21}\| \leq C\sigma'_0 \ .$$

Set $R_{21}(\xi',\eta',\beta) = Y(\xi',\eta',\beta)$ and $R_{12} = R^{t}_{21}$.

We now need to construct $R_{11}(\xi',\eta',\beta)$ in a neighborhood of $\xi' = (0,\tau')$, $\beta = 0$. To do this it is necessary to construct a matrix $D(\omega,s)$ for all $(\omega,s) \neq 0$, with $\operatorname{Re} s \geq 0$, such that

(7.8) a) $\operatorname{Re} D(\omega,s)N_{11}(\omega,s) \geq \sigma_0 I_{2p}$,

b) $u^{t}Du \geq c_0(\delta|u^{+}|^{2} - |u^{-}|^{2})$.

We need only consider (ω,s) with $\omega^2 + |s| = 1$ since we can extend $D(\omega,s)$ to all nonzero (ω,s) by

$$D(\rho\omega,\rho^2 s) = D(\omega,s) \qquad \text{for} \quad \rho > 0 \ .$$

Note that $N_{11}(\omega,s)$ is analytic in $\operatorname{Re} s \geq -\alpha_1\omega^2$. By Theorem 4.3 the eigenvalues of $N_{11}(\omega,s)$ are bounded away from the imaginary axis, so we can construct $D(\omega,s)$ locally for each value of (ω,s), $\omega^2 + |s| = 1$, using the techniques described above.

Since the set

$$\{(\omega,s):\omega^2 + |s| = 1,\ \mathrm{Re}\ s \geq 0,\ \omega \in \mathbb{R}^n\}$$

is compact, we can cover it with a finite number of neighborhoods $\{U_k\}$ on each of which $D(\omega,s)$ is defined. If $\{\varphi_k\}$ is a C^∞ partition of unity for the cover we define

$$D(\omega,s) = \sum_k \varphi_k(\omega,s)\ D(\omega,s)_{|U_k}.$$

It is easy to see that (7.8a and b) are satisfied and that $D(\omega,0)$ is a C^∞ function of (ω,s). Note that we do not define $D(0,0)$. We define

$$R(\xi',\eta',\beta) = \begin{pmatrix} dD(\omega',\beta s') & R_{21}^t(\xi',\eta',\beta) \\ R_{21}(\xi',\eta',\beta) & R_{22}(\xi',\eta') \end{pmatrix}$$

for (ξ',η',β) in a deleted neighborhood of $((0,\tau'),\eta',0)$. Note that R is not defined at $\beta = 0$, $\omega' = 0$. d is a positive number to be determined.

We now check the properties of R.

$$\mathrm{Re}\ RN' = \begin{pmatrix} \mathrm{Re}(dDN'_{11} + R_{21}^t N'_{21}) & 0 \\ 0 & \mathrm{Re}\ R_{22}N'_{22} \end{pmatrix}.$$

From (7.7) we have

$$\|R_{21}^t N_{21}'\| \le C\sigma_0'^2$$

and from (7.8)

$$\text{Re } dDN_{11}' \ge d\sigma_0' \ .$$

For d sufficiently large we have (7.4a). To check (7.4b) we note that if $\bar{v}$ is an eigenvector of N_{22}' then

$$\begin{pmatrix} 0 \\ \bar{v} \end{pmatrix}$$

is an eigenvector of N'. Also, if $\bar{u}$ is an eigenvector of N_{11}' then

$$\begin{pmatrix} \bar{u} \\ V\bar{u} \end{pmatrix}$$

is an eigenvector of N', where V is defined by

$$VN_{11}' - N_{22}'V = N_{21}' \ .$$

Since at $\beta = 0$, $\omega' = 0$ we have N_{22}' nonsingular and $N_{11}' = 0$, we can solve for $V(\xi',\eta',\beta)$ in some neighborhood of $\beta = 0$, $\omega' = 0$. Moreover, from Lemma 2.1,

$$(7.9) \qquad \|V\| \le c\|N_{21}'\| \le c\sigma_0' \ .$$

So

$$(7.10) \qquad w^+ = \begin{pmatrix} u^+ \\ v^+ + Vu^+ \end{pmatrix} \quad \text{and} \quad w^- = \begin{pmatrix} u^- \\ v^- + Vu^- \end{pmatrix} .$$

Then using (7.9),

$$w^t Rw \geq c_0(\delta(|u^+|^2 + |v^+|^2) - (|u^-|^2 + |v^-|^2)) - c\sigma_0'(|u|^2 + |v|^2)$$

$$\geq c_0'(\delta(|u^+|^2 + |v^+|^2) - (|u^-|^2 + |v^-|^2)) .$$

But by (7.10) and (7.9)

$$|w^+|^2 \leq |u^+|^2 (1 + c\sigma_0'^2) + |v^+|^2$$

$$|w^-|^2 \geq |u^-|^2 (1 - c\sigma_0'^2) + |v^-|^2 .$$

So, in a suitable neighborhood of $\beta = 0$, $\omega' = 0$, we have (7.5b).

We take especial note that the lower left $q \times 2p$ submatrix of Re RN' is zero for $|\omega'| \leq \epsilon$, $\beta' \leq \epsilon$ for some ϵ.

The final case to consider is $\beta_0 = 0$, $\omega_0' \neq 0$, $\eta_0' = 0$. By Theorem 4.3, the eigenvalues of $N_{11}'(\xi_0',0,0)$ are bounded away from the imaginary axis. As we have mentioned before, there is a matrix $U(\xi',\eta',\beta)$, analytic in a neighborhood of $(\xi_0',0,0)$ with

$$U(\xi',\eta',\beta)\, N'(\xi',\eta',\beta)\, U^{-1}(\xi',\eta',\beta) =$$

$$= N''(\xi',\eta',\beta)$$

$$= \begin{bmatrix} N''_{11} & 0 & 0 \\ N''_{21} & N''_{22} & 0 \\ 0 & 0 & N^0_{22} \end{bmatrix}$$

where the eigenvalues of $N^0_{22}(\xi'_0,0,0)$ are those eigenvalues of $N'_{22}(\xi'_0,0)$ which are pure imaginary and the eigenvalues of $N''_{22}(\xi'_0,0,0)$ are those which are not pure imaginary. Also, $N''(\xi',\eta',\beta)$ is lower triangular.

Since the eigenvalues of the upper block are bounded away from the imaginary axis, we can construct a matrix $R''(\xi'_0,0,0)$ so that

$$\mathrm{Re}\, R'' \begin{bmatrix} N''_{11} & 0 \\ N''_{21} & N''_{22} \end{bmatrix} \geq \begin{bmatrix} \sigma'_0 & 0 \\ 0 & \eta' \end{bmatrix}$$

and

$$w^t R'' w \geq c_0(\delta|w^+|^2 - |w^-|^2)$$

in a neighborhood of $\beta = 0$, $\omega'_0 = 0$, $\eta'_0 = 0$.

Let $R^0(\xi',\eta')$ be as constructed by Kreiss [6] and Ralston [15], which satisfies

$$\mathrm{Re}\, R^0(\xi',\eta')\, N^0(\xi',\eta') \geq \eta'$$

and

$$v_0^t R^0 v_0 \geq c_0(\delta|v_0^+|^2 - |v_0^-|^2) .$$

We define $R(\xi',\eta',\beta)$ by

$$R(\xi',\eta',\beta) = U^t\begin{pmatrix} R'' & 0 \\ 0 & R^0 \end{pmatrix} U .$$

As before, R satisfies (7.4) in some neighborhood of $\beta = 0$, $\omega' = 0$, $\eta' = 0$.

We now define $R(\xi,\eta)$ for all (ξ,η) as follows. The set

$$\{(\xi',\eta',\beta): 0 \leq \beta \leq \eta'/\eta_0; \ \xi'^2 + \eta'^2 = 1\}$$

is compact and we can cover it with a finite number of neighborhoods $\{U_k\}$ on each of which we have constructed $R(\xi',\eta',\beta)$. If $\{\varphi_k\}$ is a C^∞ partition of unity for this covering we define

$$\bar{R}(\xi',\eta',\beta) = \sum_k \varphi_k(\xi',\eta',\beta)\, R(\xi',\eta',\beta)_{|U_k} .$$

Then $R(\xi,\eta)$ is given by

$$R(\xi,\eta) = \bar{R}(\beta\xi,\beta\eta,\beta)$$

where $\beta = \langle\xi,\eta\rangle^{-1}$.

To see that R satisfies (7.3c) we note that for $\beta = 0$ $R(\xi',\eta',\beta)$ was constructed to be analytic in (ξ',η',β) and this would imply that $R \in Sp_1^0$. However $\bar{R}_{11}(\xi',\eta',\beta)$ is not defined at $\beta = 0$, $\xi' = (0,\tau')$, and in a neighborhood of this point $\bar{R}_{11} = dD(\omega',\beta s')$. Therefore $R_{11}(\xi,\eta) = dD(\omega,s)$ for $|\omega| \leq c_1|s|$ where $|s| \geq c_2$, and $D(\omega,s) \in Sp_{1/2}^0$ by Lemma 7.2. This means $R_{11} \in Sp_{1/2}^0$ and so $R \in Sp_{1/2}^0$.

To see that (7.3d) is satisfied we recall that for $|\omega'| \leq \epsilon, \beta' \leq$ the lower left $q \times 2p$ submatrix of $\operatorname{Re} RN'$ was constructed to be zero. This means that for

$$|\omega| \leq c_1|s|, \qquad c_2 \leq |s|$$

the lower left $q \times 2p$ submatrix of $\operatorname{Re} R(\xi,\eta)\, N(\xi,\eta)$ is zero.

In this way we can construct $R(z_0,\xi,\eta)$ for every z_0. Moreover the coefficients of $N(z,\xi,\eta)$ are continuous, so we can use the same $R(z_0,\xi,\eta)$ in a whole neighborhood of z_0. Since the coefficients of $N(z,\xi,\eta)$ tend to constants for large values of $|z|$ we can choose a finite set of points $\{z_k\}_{k=1}^M$ and neighborhoods of the points $\{U_k\}_{k=1}^M$ which cover $\{z = (y,t) : y \in \mathbb{R}^n,\ t \geq 0\}$. $R(z_k,\xi,\eta)$ satisfies (7.3) with $N(z,\xi,\eta)$ for all $z \in U_k$. If we take $\{\varphi_k\}_{k=1}^M$ to be a partition of unity for the covering, then $R(z,\xi,\eta)$ can be defined for all values of z by

$$R(z,\xi,\eta) = \sum_{k=1}^{M} R(z_k,\xi,\eta)\, \varphi_k(z).$$

7.3. Gårding Inequalities.

In this section we prove two analogues of Gårding's inequality. We then show how Theorem 4.8 follows from these results. We shall first do some rearranging of our operators to put them in a more convenient form. As in the preceding section let $z = (y,t)$, $\xi = (\omega,\tau)$, and $w = (u,v)'$ where u and v have dimensions $2p$ and q, respectively. (Thus $u = (u,\tilde{u})'$ in the notation of Chapter IV.)

We have

$$H(z,\xi,\eta) = \operatorname{Re} RN \geq c_0 \begin{pmatrix} \sigma_0 I_{2p} & 0 \\ 0 & \eta I_q \end{pmatrix}.$$

Let

$$\Lambda(\xi,\eta) = \begin{pmatrix} \sigma_0^{-1/2} I_{2p} & 0 \\ 0 & \langle \xi,\eta \rangle^{-1/2} I_q \end{pmatrix}$$

and

$$G(\xi,\eta) = \Lambda(\xi,\eta) \left[H(z,\xi,\eta) - c_0 \begin{pmatrix} \sigma_0 I_{2p} & 0 \\ 0 & \eta I_q \end{pmatrix} \right] \Lambda(\xi,\eta) . \tag{7.11}$$

Then

$$G_{11}(z,\xi,\ ) = \sigma_0^{-1} H_{11}(z,\xi,\eta) - c_0 \in Sp^0_{1/2} ,$$

$$G_{22}(z,\xi,\eta) = \langle \xi,\eta \rangle^{-1} (H_{22}(z,\xi,\ ) - c_0 \eta) \in Sp^0_1 ,$$

and

$$G_{21}(z,\xi,\eta) = \sigma_0^{-1/2} \langle \xi,\eta \rangle^{-1/2} H_{21}(z,\xi,\eta) .$$

Now $H_{21}(z,\xi,\eta) = 0$ when $\omega^2 \leq c_1(\tau^2 + \eta^2)$ for $\tau^2 + \eta^2 \geq c_2$, and for values of (ω,τ,η) with $\omega^2 \geq c_1(\tau^2 + \eta^2) \geq c_1 c_2$

$$c^{-1}\sigma_0 \leq \langle \xi,\eta \rangle \leq c\sigma_0 \; . \tag{7.12}$$

By construction, H_{21} is the product of symbols in Sp_1^0 and symbols with parabolic homogeneity. So by Lemma 7.2 and (7.12) we see that $H_{21}(z,\xi,\eta) \in Sp_1^1$ and so $G_{21}(z,\xi,\eta) \in Sp_1^0$.

We now state the first Gårding inequality we will need.

Theorem 7.2. Given the positive semidefinite matrix symbol

$$G(z,\xi,\eta) = \begin{bmatrix} G_{11}(z,\xi,\eta) & G_{21}^t(z,\xi,\eta) \\ G_{21}(z,\xi,\eta) & G_{22}(z,\xi,\eta) \end{bmatrix} \tag{7.13}$$

where $G_{11}(z,\xi,\eta) \in Sp_{1/2}^0$ and the other components are in Sp_1^0, the operator $G(z,D,\eta)$ satisfies

$$\mathrm{Re}(w,\; G(z,D,\eta)w) \geq -c_1|u|_0^2 - c_1|v|_{-1/2}^2 \; .$$

(The decomposition $w = (u,v)'$ corresponds to the matrix decomposition of (7.13).)

Proof. Our proof follows Taylor's proof of the sharp Gårding inequality in [19]. We shall modify the proof by including the parameter and shall proceed as if $G(z,\xi,\eta)$ were in Sp_1^0, and then check the estimate we obtain for $G_{11}(z,\xi,\eta)$ which is the component not in Sp_1^0.

We begin by performing a Friedrichs symmetrization of $G(z,\xi,\eta)$. Let $q(\mu)$ be an even $C_0^\infty(\mathbb{R}^{n+1})$ function with support in $|\mu| \leq 1$, and

$$\int q^2(\mu)\, d\mu = 1 .$$

Define

$$F(\xi,\zeta,\eta) = q\left(\frac{\zeta - \xi}{\langle \xi,\eta\rangle^{1/2}}\right)\langle \xi,\eta\rangle^{-(n+1)/4} .$$

Definition 7.5. The Friedrichs symmetrization of $G(z,\xi,\eta)$ is

$$S(\xi_2,z,\xi_1,\eta) = \int F(\xi_2,\zeta,\eta)\, G(z,\zeta,\eta)\, F(\xi_1,\zeta,\eta)\, d\zeta. \tag{7.14}$$

We see that $S(D,z,D,\eta)$ is a symmetric operator from

$$\begin{aligned}(w,\ S(D,z,D,\eta)w) &= \int \{(\hat{w}(\xi_2)e^{iz\cdot\xi_2}\}^t\, S(\xi_2,z,\xi_1,\eta)\,\{\hat{w}(\xi_1)e^{iz\cdot\xi_1}\}\, dz d\xi_1 d\xi_2 \\ &= \int \{\int \hat{w}(\xi_2)e^{iz\cdot\xi_2}\, F(\xi_2,\zeta,\eta)d\xi_2\}^t\, G(z,\zeta,\eta) \\ &\quad \cdot \{\int \hat{w}(\xi_1)e^{iz\cdot\xi_1}\, F(\xi_1,\zeta,\eta)d\xi_1\}\, d\zeta dz \\ &\geq 0 .\end{aligned} \tag{7.15}$$

We must now show that $S(D,z,D,\eta)$ is a pseudo-differential operator.

<u>Lemma 7.2</u>. $S(\xi_2,z,\xi_1,\eta) \in S_{p_1 1/2}^{0,0}$.

<u>Proof</u>. By induction, it can be seen that

$$(7.16) \qquad D_\xi^\alpha F(\xi,\zeta,\eta) = \langle\xi,\eta\rangle^{-(n+1)/4} \sum_{\substack{|\beta|\leq|\alpha| \\ \gamma\leq\beta}} \psi_{\alpha,\beta,\gamma}(\xi,\eta)\, \mu^\gamma D_\mu^\beta q(\mu),$$

where

$$\mu = (\xi-\zeta)\,\langle\xi,\eta\rangle^{-1/2}$$

and

$$\psi_{\alpha,\beta,\gamma}(\xi,\eta) \in S_{p_1}^{-(|\alpha|-\frac{1}{2}|\beta-\gamma|)} \subseteq S_{p_1}^{-\frac{1}{2}|\alpha|} .$$

Now

$$D_{\xi_2}^\alpha D_z^\beta D_{\xi_1}^\gamma S(\xi_2,z,\xi_1,\eta)$$

$$= \int D_{\xi_2}^\alpha F(\xi_2,\zeta,\eta)\, D_z^\beta G(z,\zeta,\eta)\, D_{\xi_1}^\gamma F(\xi_1,\zeta,\eta) d\zeta ,$$

so by the Schwarz inequality

$$|D_{\xi_2}^\alpha D_z^\beta D_{\xi_1}^\gamma S(\xi_2,z,\xi_1,\eta)|^2$$

$$\leq c_\beta \int |D_{\xi_2}^\alpha F(\xi_2,\zeta,\eta)|^2\, d\zeta \int |D_{\xi_1}^\gamma F(\xi_1,\zeta,\eta)|^2\, d\zeta .$$

Applying (7.16) we have

$$|D^{\alpha}_{\xi_2} D^{\beta}_{z} D^{\gamma}_{\xi_1} \; S(\xi_2,z,\xi_1,\eta)| \le C_{\alpha,\beta,\gamma} \langle \xi_2,\eta\rangle^{-|\alpha|/2} \, \langle \xi_1,\eta\rangle^{-|\gamma|/2}$$

which proves the lemma.

We now compare $G(z,\xi,\eta)$ and its symmetrization $S(\xi_2,z,\xi_1,\eta)$. By (7.2), $S(\xi_2,z,\xi_1,\eta)$ is equivalent to

$$T(z,\xi,\eta) \sim \sum_{\alpha \ge 0} T_{\alpha}(z,\xi,\eta)$$

where

$$T_{\alpha}(a,\xi,\eta) = i^{|\alpha|} \frac{1}{\alpha!} D^{\alpha}_{z} D^{\alpha}_{\xi_2} \; S(\xi_2,z,\xi,\eta)\big|_{\xi_2=\xi} \quad .$$

By (7.16),

$$T_{\alpha}(z,\xi,\eta) = \frac{i^{|\alpha|}}{\alpha!} \sum_{\substack{|\beta| \le |\alpha| \\ \gamma \le \beta}} \psi_{\alpha,\beta,\gamma}(\xi,\eta) \int \mu^{\gamma} D^{\beta} q(\mu)\; q(\mu)\; D^{\alpha} G(z,\xi+\mu\langle\xi,\eta\rangle^{1/2},\eta)\, d\mu \; .$$

Since $\psi_{\alpha,\beta,\gamma}(\xi,\eta) \in Sp_{1}^{-|\alpha|/2}$ and the above integral is in $Sp^{0}_{1/2}$ we have

$$\text{(7.17)} \qquad T_{\alpha}(z,\xi,\eta) \in Sp^{-1}_{1/2}$$

for $|\alpha| \ge 2$.

We now consider $T_{\alpha}(z,\xi,\eta)$ for $|\alpha| = 0$ and 1.

$$T_0(z,\xi,\eta) = \int G(z,\; \xi + \mu \; \langle\xi,\eta\rangle^{1/2},\eta)\; q^2(\mu)\; d\mu \quad .$$

Expanding G by Taylor's formula to second order the linear term integrates to zero since $q(\mu)$ is even. We have

$$T_0(z,\xi,\eta) = G(z,\xi,\eta) - \langle \xi,\eta\rangle \sum_{|\gamma|=2} G_\gamma(z,\xi,\eta) ,$$

where

$$G_\gamma(z,\xi,\eta) = \frac{1}{\gamma!} \int \mu^\gamma q^2(\mu) \int_0^1 D_\xi^\gamma G(z,\xi+\mu t \langle \xi,\eta\rangle^{1/2},\eta)(1-t)\, dt\, d\mu .$$

We now consider the components of T_0. Since $G_{11} \in Sp^0_{1/2}$

$$|T_{0,11}(z,\xi,\eta) - G_{11}(z,\xi,\eta)| \le \langle \xi,\eta\rangle \sum_{|\gamma|=2} C_\gamma \langle \xi,\eta\rangle^{-1}$$
$$\le C .$$

For the other components $G_{ij} \in Sp^0_1$, so

$$|T_{0,ij}(z,\xi,\eta) - G_{ij}(z,\xi, \ )| \le C' \langle \xi,\eta\rangle^{-1} .$$

To consider $T_\alpha(z,\xi,\eta)$ for $|\alpha| = 1$, we note that for $\gamma \le \beta$, $|\beta| = 1$,

$$\psi_{\alpha,\beta,\gamma}(\xi,\eta) \in Sp_1^{-(1-\frac{1}{2}|\beta-\gamma|)} .$$

So $\psi_{\alpha,\beta,\beta}(\xi,\eta) \in Sp_1^{-1}$, and we need only consider $\psi_{\alpha,\beta,0}(\xi,\eta)$.

$$T_\alpha(z,\xi,\eta) = \sum_{|\beta|=1} i\psi_{\alpha,\beta,0}(\xi,\eta) \int D_z^\alpha G(z,\xi+\mu\langle\xi,\eta\rangle^{1/2},\eta) D_\mu^\beta q(\mu) q(\mu) d\mu$$

plus terms in $Sp_{1/2}^{-1}$. We again expand by Taylor's formula

$$D_z^\alpha G(z,\xi+\mu\langle\xi,\eta\rangle^{1/2},\eta)$$

$$= D_z^\alpha G(z,\xi,\eta) + \langle\xi,\eta\rangle^{1/2} \sum_{|\gamma|=1} i \int_0^1 D_\xi^\gamma D_z^\alpha G(z,\xi+\mu t\langle\xi,\eta\rangle^{1/2},\eta)\mu^\gamma\, dt.$$

Inserting this in the formula for $T_\alpha(z,\xi,\eta)$, the first term integrates to zero since $q(\mu)$ is an even function. Considering the components we have

$$T_{\alpha,11}(z,\xi,\eta) \in Sp_{1/2}^{-1/2}$$

and for the other components

$$T_{\alpha,ij}(z,\xi,\eta) \in Sp_1^{-1}.$$

Combining all of the above we have

$$G(z,\xi,\eta) = T(z,\xi,\eta) + \begin{pmatrix} \tilde G_{11}(z,\xi,\eta) & 0 \\ 0 & 0 \end{pmatrix} + \tilde{\tilde G}(z,\xi,\eta) \tag{7.18}$$

where $\tilde G_{11}(z,\xi,\eta) \in Sp_{1/2}^0$ and $\tilde{\tilde G}(z,\xi,\eta) \in Sp_{1/2}^{-1}$.

(7.18) and (7.15) imply

$$\begin{aligned} \operatorname{Re}(w,Gw) &= \operatorname{Re}(w,Tw) + \operatorname{Re}(u,\tilde G_{11}u) + \operatorname{Re}(w,\tilde{\tilde G}w) \\ &\geq -c_1|u|_0^2 - c_2|w|_{-1/2}^2 \end{aligned} \tag{7.19}$$

which proves Theorem 7.2.

We next prove another Gårding inequality.

Theorem 7.3. If $P(z,\xi,\eta) \in Sp_\rho^0$ and $P(z,\xi,\eta)^t = P(z,\xi,\eta) \geq c_0 > 0$, then for each $\epsilon > 0$, $r > 0$ there is a constant $c_r^\epsilon > 0$ such that

$$\operatorname{Re}(w,P(z,D,\eta)w) \geq (c_0 - \epsilon)\,|w|_0^2 - c_r^\epsilon |w|_{-r}^2 . \tag{7.20}$$

Proof. We modify Taylor's proof in [19, p.38] to allow matrix symbols. We show that we can write

$$\operatorname{Re} P = \frac{1}{2}(P + P^*) = c_0 - \epsilon + D_r^* D_r + P_r$$

where $P_r \in Sp_\rho^{-r\rho}$ and $D_r \in Sp_\rho^{-(r-1)\rho}$. Since $P(z,\xi,\eta) \in Sp_\rho^0$, its adjoint $P^*(z,\xi,\eta)$ is also in Sp_ρ^0 and by (7.1)

$$P^*(z,\xi,\eta) = P(z,\xi,\eta) + \tilde{P}(z,\xi,\eta)$$

where $\tilde{P} \in Sp_\rho^{-\rho}$.

Let

$$P_0(z,\xi,\eta) = \frac{1}{2}\left(P(z,\xi,\eta) + P^*(z,\xi,\eta)\right) + M\,\langle\xi,\eta\rangle^{-\rho} - (c_0 - \epsilon)$$

where M is chosen so that

$$P_0(z,\xi,\eta) \geq \epsilon . \tag{7.21}$$

Note that $P_0^* = P_0$ and $P_0^t = P_0$. Let $B_0(z,\xi,\eta) = P_0(z,\xi,\eta)^{1/2}$, by (7.21) $B_0 \in Sp_\rho^0$ and since $B_0^t = B_0$

$$B_0^*(z,\xi,\eta) - B_0(z,\xi,\eta) \in Sp_\rho^{-\rho}.$$

So

$$\frac{1}{2}(P + P^*) = c_0 - \epsilon + B_0^* B_0 + P_1$$

where

$$P_1(z,\xi,\eta) = P_1^*(z,\xi,\eta) \in Sp_\rho^{-\rho}.$$

Now assume that we have constructed $B_j(z,\xi,\eta) \in Sp_\rho^{-j\rho}$ for $0 \le j < n$ such that

$$\frac{1}{2}(P + P^*) = c_0 - \epsilon + (B_0^* + \cdots + B_{n-1}^*)(B_0 + \cdots + B_{n-1}) + P_n \tag{7.22}$$

where $P_n(z,\xi,\eta) \in Sp_\rho^{-n\rho}$ and $P_n^* = P_n$.

We take $B_n(z,\xi,\eta)$ as the solution to

$$B_0 B_n + B_n B_0 = P_n .$$

Since $B_0 \ge \sqrt{\epsilon}$, $B_n(z,\xi,\eta)$ is a well-defined matrix, and because $B_0 \in Sp_\rho^0$ and $P_n \in Sp_\rho^{-n\rho}$ we have $B_n \in Sp_\rho^{-n\rho}$. Also since $P_n^* = P_n$,

$$\begin{aligned} &B_0(B_n^* - B_n) + (B_n^* - B_n)B_0 \\ &\quad = (B_0 - B_0^*)B_n^* + B_n^*(B_0 - B_0^*) \in Sp_\rho^{-(n+1)\rho} \end{aligned}$$

and so $B_n^* - B_n \in Sp_\rho^{-(n+1)\rho}$. Now it is easy to see that (7.22) holds if n is replaced by $n+1$. Let $D_n = B_0 + \cdots + B_{n-1}$, then from (7.22) we have

$$\begin{aligned} \mathrm{Re}(w,Pw) &= \frac{1}{2}((w,Pw) + (w,P^*w)) \\ &= (c_0 - \epsilon)\,|w|_0^2 + |D_n w|^2 + (w,P_n w) \\ &\geq (c_0 - \epsilon)\,|w|_0^2 - c_n|w|_{-n\rho/2}^2 \quad . \end{aligned}$$

Taking $(n\rho)/2 \geq r$ we have proved Theorem 7.3.

7.4. Proof of Theorem 4.8.

In this section we prove Theorem 4.8. First we relate the norms used in Chapter IV to those used in this chapter.

For two vector-valued functions u and v

$$(u,v)_\eta = (e^{-\eta t}u,\ e^{-\eta t}v) \ ,$$

and for a pseudo-differential operator P, $(w,Pw)_\eta$ will mean

$$(e^{-\eta t}w,\ P(e^{-\eta t}w)) \quad .$$

(So to be accurate we should write

$$(w,\ e^{\eta t}P(e^{-\eta t}w))_\eta$$

in place of $(w,Pw)_\eta$.)

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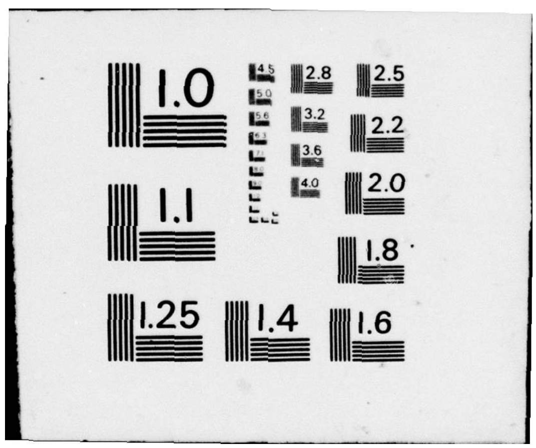
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We now prove Theorem 4.8. Applying Theorem 7.2 and using (7.11) we have

$$
\begin{aligned}
&\mathrm{Re}(w, R(z,D,\eta)\, N(z,D,\eta) w)_\eta \\
&\quad \geq c_0 \|\Lambda(D,\eta)^{-1} w\|_\eta^2 - c_1 \|\sigma_0^{1/2} u\|_\eta^2 - c_2 \|v\|_\eta^2 \\
&\quad \geq (c_0 - c_1)\, \mathrm{Re}(u, \Sigma u)_\eta + (\eta c_0 - c_2)\, \|v\|_\eta^2 \;.
\end{aligned}
$$

By taking η sufficiently large we can make the coefficient of $\|v\|_\eta^2$ larger than $\frac{1}{2} c_0 \eta$. To handle the other term on the right we note that we can replace $R(z,\xi,\eta)$ by

$$
\begin{pmatrix} (d+1)R_{11} & R_{21}^t \\ R_{21} & R_{22} \end{pmatrix}
$$

where $d > 0$, and all the properties of $R(z,\xi,\eta)$ still hold. Moreover, $\mathrm{Re}\, R_{11}(z,\xi,\eta)\, N_{11}(z,\xi,\eta)\, \sigma_0^{-1} \geq c_0$ so we can apply Theorem 7.3 to obtain

$$
\begin{aligned}
&\mathrm{Re}(u, R_{11}(z,D,\eta)\, N_{11}(z,D,\eta) u)_\eta \\
&\quad \geq (c_0 - \epsilon - c\eta^{-1})\, \mathrm{Re}(u, \Sigma u)_\eta \quad .
\end{aligned}
$$

For proper choice of d we then have

$$
\mathrm{Re}(w, RN\, w)_\eta \geq c\{\mathrm{Re}(u, \Sigma u)_\eta + \eta \|v\|_\eta^2\} \quad .
$$

To change notation to that of Chapter IV replace u by $(u, \Sigma^{-1}(u_x + P_0^{-1}Av))$ and we easily obtain (4.28).

To obtain (4.29) we note that for any vectors w and g such that

$$T(z,\xi,\eta)w = g$$

we have

$$w^t R(z,\xi,\eta)w \geq -c_1|g|^2 + c_2|w|^2 .$$

But this is equivalent to

$$R + c_1 T^t T \geq c_2 .$$

From the definition of $T(z,\xi,\eta)$ in (4.7) and Lemma 7.1 we have $T \in Sp^0_{1/2}$. So we may apply Theorem 7.3 and we have

$$\begin{aligned} \text{Re } \langle w, R(z,D,\eta)w\rangle_\eta &\geq -c_1 \text{Re}\langle w, T^t T w\rangle_\eta + (c_2 - \epsilon - \tfrac{1}{\eta}) \; |w|^2_\eta \\ &\geq -c_1 |Tw|^2_\eta + (c_2 - \epsilon - \eta^{-1} - c\eta^{-1/2}) \; |w|^2_\eta . \end{aligned}$$

We have used

$$T^{t*}(z,\xi,\eta) - T(z,\xi,\eta) \in Sp^{-1/2}_{1/2}$$

from (7.1). For η sufficiently large we have (4.28).

This completes the proof of Theorem 4.8.

APPENDIX

In this appendix we present a few examples to illustrate the use of the results of this thesis. The first four examples treat constant coefficient problems on a half-space. The fifth example illustrates how to deal with variable coefficient problems on a bounded domain with smooth boundary.

For each example, the quantities u, v, v^1, v^2 will be scalars and for the half-space we will take $\mathbb{R}^2_+ = \{(x,y): x \geq 0,\ y \in \mathbb{R}\}$.

Example 1. The equations are

$$\begin{aligned} u_t &= u_{xx} + u_{yy} + av_y \\ v_t &= bu_x + \phantom{u_{yy} + a}v_x . \end{aligned}$$

The boundary condition is

$$k_1 u + k_2 v = g(y,t) .$$

We will use Theorem 4.5 to establish well-posedness. Note that we have one boundary condition since $p + q^- = 1 + 0 = 1$ (Assumption 4.5).

a) We first check for parabolic eigensolutions (Definition 4.5). If (u, v_0) is a parabolic eigensolution, then $v_0 = 0$ since $q^- = 0$. The equation for u is

$$su = u_{xx} - \omega^2 u ,$$

and so

$$u = u_0 e^{-\sigma x} , \qquad \sigma = (\omega^2 + s)^{1/2} .$$

u_0 must be non-zero and satisfy

$$k_1 u_0 + k_2 \cdot 0 = 0$$

if it is an eigensolution. We see that there are no eigensolutions of parabolic type if and only if $k_1 \neq 0$.

b) We now check for eigensolutions of hyperbolic type (Definition 4.6). If (u,v) is a hyperbolic eigensolution, it satisfies

$$0 = u_{xx} - \omega^2 u$$

$$sv = bu_x + v_x .$$

The $L^2(\mathbb{R}_+)$ solutions are

$$u = u_0 e^{-|\omega| x}$$

$$v = - \frac{b|\omega|}{s + |\omega|} u_0 e^{-|\omega| x} .$$

The boundary condition is

$$k_1 u_0 + k_2 \left(- \frac{b|\omega|}{s + |\omega|} u_0\right) = 0 .$$

Assuming that $k_1 \neq 0$, we have

$$s = \left(\frac{k_2}{k_1} b - 1 \right) |\omega| \ .$$

$\operatorname{Re} s \geq 0$ if and only if $\operatorname{Re}(k_2/k_1)b \geq 1$, and so the system is σ-well-posed if and only if $\operatorname{Re}(k_2/k_1)b < 1$, with $k_1 \neq 0$.

<u>Example 2</u>. $$u_t = u_{xx} + u_{yy}$$

$$\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}_t = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}_x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}_y + \begin{pmatrix} 0 \\ b \end{pmatrix} u_x$$

with boundary conditions

$$v^1 = g^1(y,t)$$

$$u + kv^2 = g^2(y,t) \ .$$

We will use Theorem 4.5 again and note that Assumption 4.1 is satisfied since $p = 1$ and $q^- = 1$.

a) If (u, v_0) is an eigensolution of parabolic type, then

$$v_0 = \begin{pmatrix} v_0^1 \\ 0 \end{pmatrix},$$

and u satisfies

$$su = u_{xx} - \omega^2 u \ .$$

We have $u = u_0 e^{-\sigma x}$ and the boundary conditions are

$$v_0^1 = 0$$

$$u_0 + k \cdot 0 = 0 \; .$$

There are obviously no parabolic eigensolutions.

b) An eigensolution of hyperbolic type must solve:

$$0 = u_{xx} - \omega^2 u$$

$$s \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}_x + \begin{pmatrix} 0 & i\omega \\ i\omega & 0 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix} u_x \; .$$

It is easily seen that the solutions are:

$$u = u_0 e^{-|\omega| x},$$

$$\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = v_0 \begin{pmatrix} 1 \\ \frac{i\omega}{s+\lambda} \end{pmatrix} e^{-\lambda x} + \frac{|\omega| b}{s^2} u_0 \begin{pmatrix} i\omega \\ s-|\omega| \end{pmatrix} e^{-|\omega| x}$$

for $s \neq 0$, and when $s = 0$,

$$\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = v_0 \begin{pmatrix} 1 \\ i \frac{\omega}{|\omega|} \end{pmatrix} e^{-|\omega| x} + \frac{b}{2} u_0 \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} i\omega \\ -|\omega| \end{pmatrix} x \right] e^{-|\omega| x}.$$

We have set $\lambda = (\omega^2 + s^2)^{1/2}$. The boundary condition for $s \neq 0$ is

$$v_0 + \frac{i\omega|\omega|b}{s^2} u_0 = 0$$

$$u_0 + k\left(\frac{i\omega}{s+\lambda} v_0 + \frac{b|\omega|(s-|\omega|)}{s^2} u_0\right) = 0 .$$

Solving for s, we have

$$\sqrt{(s^2 + \omega^2)} + |\omega| + |\omega|kb = 0 ,$$

or, since $\omega \neq 0$,

$$\sqrt{(\frac{s}{|\omega|})^2 + 1} = -(1 + kb) .$$

We obtain the same equation for $s = 0$. The map $z \rightarrow \sqrt{z^2 + 1}$ maps the plane $\text{Re } z \geq 0$ onto itself, so we have that the system is σ-well-posed if and only if

$$\text{Re}(1 + kb) > 0.$$

We now present two examples illustrating the use of Theorem 5.1 for strongly σ-well-posed systems.

<u>Example 3</u>. We take the equations

$$u_t = u_{xx} + u_{yy} + av_y$$
$$v_t = bu_x \qquad - v_x$$

with boundary conditions

$$u_x + h_1 v = g_1(y,t)$$
$$u_y + h_2 v = g_2(y,t) \ .$$

Note that Assumption 4.1 is satisfied.

a) If (u, v_0) is a strong parabolic eigensolution (Definition 5.2), then

$$u = u_0 e^{-\sigma x} \ , \qquad (\sigma = (\omega^2 + s)^{1/2})$$

and

$$-\sigma u_0 + h_1 v_0 = 0$$
$$i\omega u_0 + h_2 v_0 = 0 \ .$$

If $h_2 = 0$ we have an eigensolution with $\omega = 0$ and $\sqrt{s} = h_1 v_0 / u_0$. If $h_2 \neq 0$,

$$s = -\omega^2 \{1 + (\frac{h_1}{h_2})^2\} \ .$$

We see that there are no strong eigensolutions of parabolic type if and only if

$$h_2 \neq 0$$
$$\mathrm{Re}(\frac{h_1}{h_2})^2 > -1 \ .$$

b) Checking for strong eigensolutions of hyperbolic type (Definition 5.3), we solve

$$0 = u_{xx} - \omega^2 u + ai\omega v$$
$$sv = -v_x \ .$$

The solutions are

$$v = v_0 e^{-sx}$$

$$u = u_0 e^{-|\omega| x} - \frac{ai\omega}{s^2 - \omega^2} v_0 e^{-sx}$$

when $s \neq |\omega|$, and

$$u = u_0 e^{-|\omega| x} + \frac{ai\omega}{2|\omega|} v_0 x e^{-|\omega| x}$$

when $s = |\omega|$.

The boundary conditions are satisfied if

$$\frac{s}{|\omega|} = \frac{a}{h_2 \pm ih_1} - 1 .$$

The "$\pm$" sign depends on the sign of ω. The system is strongly σ-well-posed if and only if all of the following are satisfied

$$h_2 \neq 0 ,$$

$$\mathrm{Re}\left(\frac{h_1}{h_2}\right)^2 > -1,$$

$$\mathrm{Re}\,\frac{a}{h_2 - ih_1} < 1 ,$$

and

$$\mathrm{Re}\,\frac{a}{h_2 - ih_1} < 1 .$$

Example 4. We take the same equations as in Example 2.

$$u_t = u_{xx} + v_{yy}$$

$$\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}_t = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}_x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}_y + \begin{pmatrix} 0 \\ b \end{pmatrix} u_x ,$$

but with the boundary conditions

$$u_x + h_1 v^1 = g^1$$
$$u_y + h_2 v^2 = g^2 .$$

It is easy to see that the condition that no strong parabolic eigensolutions exist is $h_1 \neq 0$.

If (u,v) is a strong hyperbolic eigensolution,

$$u = u_0 e^{-|\omega| x}$$

$$\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = v_0 \begin{pmatrix} 1 \\ \frac{i\omega}{s+\lambda} \end{pmatrix} e^{-\lambda x} , \qquad (\lambda = (\omega^2 + s^2)^{1/2}) .$$

The boundary condition is satisfied if

$$s + \lambda = \frac{h_2}{h_1} |\omega| ,$$

or

$$\frac{s}{|\omega|} + \sqrt{\left(\frac{s}{|\omega|}\right)^2 + 1} = \frac{h_2}{h_1} .$$

The map $z \longrightarrow w = z + \sqrt{z^2 + 1}$ maps the plane $\mathrm{Re}\, z \geq 0$ onto the region $\mathrm{Re}\, w \geq 0$, $|w| \geq 1$. Thus the system is strongly σ-well-posed if and only if

$$h_1 \neq 0 \,,$$

and

$$\mathrm{Re}\, \frac{h_2}{h_1} < 0$$

or

$$\left|\frac{h_2}{h_1}\right| < 1.$$

For our fifth example we apply the results of Examples 1 and 3.

Example 5. We take Ω to be an annulus in $\mathbb{R}^2$,

$$\{(r,\theta): 0 < r_0 \leq r \leq r_1,\ 0 \leq \theta \leq 2\pi\}$$

and our equations are

$$u_t = \Delta u + \frac{a(r,\theta,t)}{r} \frac{\partial v}{\partial \theta}$$

$$v_t = -b(r,\theta,t) \frac{\partial u}{\partial r} - \frac{\partial v}{\partial r} \,.$$

For our boundary conditions we take for $r = r_0$,

$$u_r + h_1(\theta,t)v = g_1^0(\theta,t)$$

$$\frac{1}{r_0} u_\theta + h_2(\theta,t)v = g_2^0(\theta,t) \ ,$$

and for $r = r_1$

$$k_1(\theta,t)u + k_2(\theta,t)v = g^1(\theta,t).$$

We assume that all coefficients tend to constants as $t \to \infty$. Note that on $r = r_1$, $\frac{\partial}{\partial r}$ becomes $-\frac{\partial}{\partial x}$ in the notation of Definition 6.1.

From Examples 1 and 3 we have sufficient conditions for the system to be well-posed:

$$k_1(\theta,t) \neq 0 \ ,$$

$$\operatorname{Re}\left[\frac{k_2(\theta,t)}{k_1(\theta,t)} b(r_1,\theta,t)\right] < 1 \ ,$$

$$h_2(\theta,t) \neq 0 \ ,$$

$$\operatorname{Re}\left[\left(\frac{h_1(\theta,t)}{h_2(\theta,t)}\right)^2\right] > -1 \ ,$$

$$\operatorname{Re}\left[\frac{a(r_0,\theta,t)}{h_2(\theta,t) + ih_1(\theta,t)}\right] < 1 \ ,$$

$$\operatorname{Re}\left[\frac{a(r_0,\theta,t)}{h_2(\theta,t) - ih_1(\theta,t)}\right] < 1,$$

and these inequalities must also be satisfied in the limit as $t \to \infty$.

Using the techniques of Chapter VI we obtain the estimate

$$\|u\|^2_{\eta,\Omega} + \|v\|^2_{\eta,\Omega} + \|\nabla u\|^2_{\eta,\Omega_\epsilon} + |u|^2_{\eta,\partial\Omega} + |v|^2_{\eta,\partial\Omega} + |\nabla u|^2_{\eta,r=r_0}$$

$$\leq C(|g^0_1|^2_{\eta,r=r_0} + |g^0_2|^2_{\eta,r=r_0} + |g^1|^2_{\eta,r=r_1}),$$

where

$$\Omega_\epsilon = \{(r,\theta): r_0 \leq r < r_1 - \epsilon\}.$$

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This thesis deals with initial boundary value problems for incompletely parabolic systems of partial differential equations. Such systems can be described as a second order Petrovskii parabolic system and a first order hyperbolic system coupled together by terms with first order spatial derivatives. The dependent variables are then of either parabolic or hyperbolic type. Examples are the equations for couple sound and heat flow and the viscous shallow water equations.

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The Cauchy problem for incompletely parabolic systems is well-posed and by means of the energy method we prove existence and uniqueness of the solution.

The method we use for the initial boundary value problem is similar to that used by Kreiss for strictly hyperbolic systems. We first treat systems with constant coefficients on a half-space. The boundary conditions are linear combinations of the dependent variables and many contain first derivatives of the variables of parabolic type. We obtain necessary and sufficient conditions for the initial boundary value problem to be well-posed when the coefficients are constant and the domain is a half-space. The appropriate norm involves weighting factors which accomodate both hyperbolic and parabolic variables.

The well-posedness is equivalent to the non-existence of eigensolutions. Eigensolutions are solutions to simpler problems than the original incompletely parabolic initial boundary value problem. Eigensolutions are of either parabolic or hyperbolic type and their presence indicates that the solution does not depend continuously on the boundary data in the appropriate norm. The conditions for the non-existence of eigensolutions are essentially algebraic in character.

We also consider incompletely parabolic systems with smooth coefficients on bounded domain with smooth boundary. We show that the initial boundary value problem is well-posed if certain constant coefficient problems on half-spaces are well-posed. This is carried out using freezing arguments which utilize Gårding's inequalities for pseudo-differential operators.

In the final chapter we develop a theory of pseudo-differential operators that depend on a parameter. This theory is used to prove the necessary Gårding's inequalities referred to above.

In an appendix we present some illustrations of the use of the methods presented in this thesis.